On the convergence of certain infinite processes to rational numbers

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This are notes on my theorems on the topic. The full paper was disseminated to a large group of mathematicians in 2003 .

Remark 1: In my paper the following or similar statement often appears: "Let $f(n)$ be a function which is rational for all integers n." By this I do not mean that the function f is a quotient of polynomials in n. What I mean if that f takes on rational values for all integers n. Thus we could have $f(n) = \frac{3^{n-7}}{n!}$ or $f(n) = n^n$ or $f(n) = \frac{3n^2-1}{n^4+2}$. I realize that the term "rational function" generally means a quotient of polynomials. However I do not use the term in that sense anywhere in the paper. When I refer to functions which are quotients of polynomials I state that the function is a "quotient of polynomials".

Consider the function $f(n) = \frac{p(n)}{q(n)}$, where p and q are polynomials of n with rational coefficients. Then break $f(n)$ into partial functions. Partial fractions must be made by factorizing the denominator $q(n)$ into polynomials with rational coefficients such that the partial fractions cannot be further broken into partial fractions. In each partial fraction the degree of the numerator should be less than the degree of the denominator; further, the numerator should have only one term. Suppose , breaking into partial fractions we get $f(n) = \frac{1}{(2n-1)} - \frac{1}{(2n)}$. Then we can split one or more of these original partial fractions to get, say, $f(n) = \frac{1}{(2n-1)} - \frac{1}{(n)} + \frac{1}{(2n)}$. For theorem 2 it is important to note that partial fractions can be split up to get a new set of fractions, as shown in the above example. In theorem 2 it may happen that the original break up into partial fractions does not satisfy the equality mentioned in the theorem, but by using this method of "part splitting" we may get a different set of fractions that do satisfy the equality. "Part-splitting" results simply from writing a rational number as the sum of two or more rational numbers.

The term "translation" has the usual meaning. Let $f(n) = f_1(n) + \cdots + f_j(n)$. Then the term "translation" means replacing $f_i(n)$, for some $i(1 \leq i \leq j)$, with $f_i(n + c_i)$, where c_i is an integer.

Let $f(n) = f_1(n)+\cdots+f_j(n)$. Then define the term "increase relative occurrence of part" to mean replacing $f_i(n)$, for some $i(1 \leq i \leq j)$, with $(f_i(kn)+f_i(kn-1)+\cdots+f_i(kn-(k-1))$, where k is a positive integer. To do the reverse replacement is to "decrease relative occurrence of part." Suppose we increase and/or decrease the relative occurrence of part p times. For each qth time $1 \leq q \leq p$ we increase relative occurrence of some part $f_i(n)$ we will need to counter this replacement by the function $D_q(n) = [-f_i(n+1) - f_i(n+2) - \cdots - f_i(kn)]$. Note that $D_q(n) = \sum_{r=1}^{n} [f_i(r) - (f_i(kr) + f_i(kr-1) + \cdots + f_i(kr-(k-1))]$. For each qth time $1 \leq j \leq p$

we decrease relative occurrence of part we will need $D_q(n) = [f_i(n+1) + f_i(n+2) + \cdots + f_i(kn)]$. q simply keeps count of how many times we have used increasing or decreasing relative occurrence of part; because q is just an independent counter it appears only on the left side in the above equation and not on the right. Define $D(n) = D_1(n) + D_2(n) + \cdots + D_p(n)$, where each D_q is gotten by one of the above two formulas. Examples: $f(n) = \frac{1}{n} + \frac{1}{n^2}$. Increase relative occurrence of part to get $f(n) = \frac{1}{3n} + \frac{1}{3n-1} + \frac{1}{3n-2} + \frac{1}{n^2}$ and to get $D_1(n) = -\frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{3n}$. Going from the second $f(n)$ to the first would be to decrease relative occurrence of part.

In theorem 2, if on breaking $f(n)$ into partial fractions we get partial fractions of the same degree but having different coefficients associated with the powers of n, then this increased and decreased relative occurrence of part may make it possible to rearrange the partial fractions so that they satisfy the equality mentioned in theorem 2.

Theorem 2: Let $f(n)$ be a function which is either always positive or always negative for all integers $n > N$, where N is a positive integer. Further let $f(n) = \frac{p(n)}{q(n)}$, where p and q are polynomials of n with rational coefficients and let $\sum_{n=1}^{\infty} f(n)$ be convergent series. Break $f(n)$ into partial fractions, to get $f(n) = u_1(n) + \cdots + u_q(n)$. Rearrange these partial fractions by increasing or decreasing the relative occurrence of parts and by part-splitting and by translation to get a new set of fractions $f_1(n), \ldots, f_j(n)$. Then if $f_1(n) + \cdots + f_j(n) = 0$ then the infinite series is rational or irrational depending on whether $\lim_{n\to\infty} D(n)$ is rational or irrational.

If we cannot obtain $f_1(n), \ldots, f_j(n)$ such that $f_1(n) + \cdots + f_j(n) = 0$ then the infinite series is irrational.

Note that we can rearrange by using the above three methods in any order and as many times as needed.

Examples for Theorem 2:

We wish to establish whether the convergent series $\sum_{n=1}^{\infty} f(n)$ is rational or irrational for the below values of $f(n)$.

It can be seen that, if by using the three methods in any order and as many times as needed we can get $f_1(n), \ldots, f_j(n)$ such that $f_1(n) + \cdots + f_j(n) = 0$, then we can also obtain the same equality by the following procedure: first use only part-splitting, and then use only translation. The new set of fractions obtained should be such that we can just use increasing relative occurrence of parts to achieve the equality (and given a set of fractions it can be quickly seen if it is possible to rearrange them by only increasing relative occurrence of part so as to obtain the required equality).

So we can just use the above procedure and thus have an easy method of checking.

Also note that to obtain $f_1(n), \ldots, f_j(n)$ translation *alone* will generally do for series connected with important functions, and it is rare that part-splitting and increasing or decreasing relative occurrence of part will even come into play.

$f(n) = \frac{1}{(1+6n)(5+6n)}$. Taking partial fractions we get $f(n) = u_1(n) + u_2(n)$, where $u_1(n) = \frac{1}{4(1+6n)}$ and $u_2(n) = \frac{-1}{4(5+6n)}$. There exist no integers i_1, i_2 such that $u_1(n+i_1) + u_2(n+i_2) = 0$. Using part-splitting and then translation will not enable us to get a set of fractions which will satisfy the required equality by increasing relative occurrence of parts; this is because in both denominators n has the same coefficient of 6. So series irrational.

$f(n) = \frac{1}{(1+6n)(7+6n)}$. Taking partial fractions we get $f(n) = u_1(n) + u_2(n)$, where $u_1(n) = \frac{1}{6(1+6n)}$ and $u_2(n) = \frac{-1}{6(7+6n)}$. $u_1(n+1) + u_2(n+0) = 0$. So series rational.

$f(n) = \frac{1}{(2n+3)(2n+1)(2n-1)}$. Taking partial fractions we get $f(n) = u_1(n) + u_2(n) + u_3(n)$, where $u_1(n) = \frac{1}{8(2n+3)}, u_2(n) = \frac{-1}{4(2n+1)}$ and $u_3(n) = \frac{1}{8(2n-1)}$. $u_1(n+0) + u_2(n+1) + u_3(n+2) = 0$. So series rational.

$f(n) = \frac{1}{(2n+1)(2n)(2n-1)}$. Taking partial fractions we get $f(n) = u_1(n) + u_2(n) + u_3(n)$, where $u_1(n) = \frac{1}{2(2n+1)}, u_2(n) = \frac{-1}{(2n)}$ and $u_3(n) = \frac{1}{2(2n-1)}$. There exist no integers i_1, i_2, i_3 such that $u_1(n+i_1) + u_2(n+i_2) + u_3(n+i_3) = 0$. We could try part-splitting using $\frac{-1}{(2n)} = \frac{-1}{(n)} + \frac{-1}{(2n)}$. And then increase relative occurrence of $\frac{-1}{(n)}$ by replacing it with $\frac{-1}{(2n)} + \frac{-1}{(2n-1)}$. We could try this with the other partial fractions too. However, it can be seen that using part-splitting and then translation will not enable us to get a set of fractions which will satisfy the required equality by increasing relative occurrence of parts. So series irrational.

$f(n) = \frac{1}{n^k}, k$ is an integer and $k > 1$. Can't be broken into partial fractions. Using part-splitting and then translation will not enable us to get a set of fractions which will satisfy the required equality by increasing relative occurrence of parts. So series irrational.

$f(n) = \frac{(-1)^n}{n}$. The theorem applies only to series with all terms positive or negative and does not apply to alternating series. Noting that the series starts at $n = 1$ and pairing every two terms of the alternating series we get a new series equal to the alternating series. New series is $g(n) = \frac{-1}{2n-1} + \frac{1}{2n}$. We get $g(n) = u_1(n) + u_2(n)$, where $u_1(n) = \frac{-1}{2n-1}$ and $u_2(n) = \frac{1}{2n}$. There exist no integers i_1, i_2 such that $u_1(n+i_1)+u_2(n+i_2) = 0$. However, by part-splitting we get a new set of fractions: $u_1(n) = \frac{1}{n}, u_2(n) = \frac{-1}{2n}$ and $u_3(n) = \frac{-1}{2n-1}$. Increasing the relative occurrence of $\frac{1}{n}$ we replace it by $\frac{1}{2n} + \frac{1}{2n-1}$. We get $u_1(n) = \frac{-1}{2n-1}, u_2(n) = \frac{-1}{2n}, u_3(n) = \frac{1}{2n}$ and $u_4(n) = \frac{1}{2n-1}$.$u_1(n+0) + u_2(n+0) + u_3(n+0) + u_4(n+0) = 0$. We have $D(n) = [-\frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n}]$. $\lim_{n\to\infty} D(n)$ is known to be irrational. So series irrational.

$f(n) = \frac{3}{4n^2} - \frac{1}{(2n+3)^2}$. We get $f(n) = u_1(n) + u_2(n)$, where $u_1(n) = \frac{3}{4n^2}$ and $u_2(n) = \frac{-1}{(2n+3)^2}$. There exist no integers i_1, i_2 such that $u_1(n + i_1) + u_2(n + i_2) = 0$. However, by part-splitting we get a new set of fractions: $u_1(n) = \frac{1}{n^2}, u_2(n) = \frac{-1}{(2n)^2}$ and $u_3(n) = \frac{-1}{(2n+3)^2}$. Increasing the relative occurrence of $\frac{1}{n^2}$ we replace it by $\frac{1}{(2n)^2} + \frac{1}{(2n-1)^2}$. We get $u_1(n) = \frac{1}{(2n)^2}, u_2(n) = \frac{1}{(2n-1)^2}, u_3(n) = \frac{-1}{(2n)^2}$ and $u_4(n) = \frac{1}{(2n+3)^2}$.$u_1(n+0) + u_2(n+0) + u_3(n+0) + u_4(n-2) = 0$. We have $D(n) = [-\frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} - \cdots - \frac{1}{(2n)^2}]$. $\lim_{n\to\infty} D(n)$ can be seen to converge to the rational number 0. So series rational.

Theorem 3: Let $f(n)$ be a function which is rational for all positive integers n and which is either always positive or always negative for all integers $n > N$, where N is a positive integer. Then the series $\sum_{n=1}^{\infty} f(n)$ converges to a rational number if and only if there exists a function $F(n)$, such that

(1)$F(n)$ is rational for all positive integers n.

(2)$f(n) = F(n) - F(n+1)$ and

(3) $\lim_{n\to\infty} F(n) = 0$

(4) $F(n)$ must be expressible in a form other than an infinite series.

(5) $\lim_{n\to\infty} F(n)$ must be expressible in a form other than an infinite series.

"Expressible in a form other than an infinite series" means that when we write the expression for $F(n)$ or for $\lim_{n\to\infty} F(n)$, the expression must not contain an infinite series.

Remark 4: What does theorem 3 tell us? It states that in all series with rational terms that converge to a rational number, $f(n)$ can be broken into parts which cancel each other out. The added conditions (4) and (5) serve to eliminate choices where $F(n)$ is a circular or "fake" where $f(n)$ is not being "actually" broken into two parts, even though we have $f(n) = F(n) - F(n+1)$. Without these conditions theorem 3 would be unusable and would just be a circular statement. With these conditions theorem 3 becomes a powerful statement which can be used to examine the question of convergence to rational or irrational numbers infinite series. For example, it can be proved that theorem 2 follows from theorem 3. *Instead* of conditions (4) and (5) we could say that $F(n)$ must be a "closed expression" or a "closed form"; instead of (4) and (5) we could also say that the infinite series $\sum_{n=1}^{\infty} f(n)$ is a "telescoping series".

Remark 5: We can also use the ideas of these methods to analyze infinite series, and learn facts about them other than what kind of real number they converge to. Let us look at the famous binomial series which is associated with Newton and Abel. The following is from a 1826 paper by Neils H. Abel:

"One of the most remarkable series of algebraic analyses is the following: $(1+x)^m = 1+\frac{m}{1}x+\frac{m(m-1)}{1\cdot 2}x^2+\cdots$ etc.

$\cdots$ It is assumed that the numerical equality will always occur whenever the series is convergent, but this has never yet been proved."

Abel finally proved this. Now an alternate way to prove this would be that if we knew the infinite series converges to a rational number then there must exist a cancellation pattern. A search for a cancellation pattern, say for the case $m = \frac{1}{2}$ leads to replacing x by $y^2 - 1$. Also further replace $y^2 - 1 = (y-1)(y+1)$ by $z(z-2)$, putting $z = 1 - y$. In each term replacing x with $z^2 - 2z$, expanding each term, and adding the same powers of z from all terms, gives us the required cancellation pattern. When $m = \frac{1}{3}$ we would begin by replacing x with $y^3 - 1$ etc. Thus this numerical equality is proved by finding and using this cancellation pattern, without actually using any of the theorems in this paper. This example is unusual because here we have a cancellation pattern for both cases — when the series converges to a rational or to an irrational number; this kind of occurrence is what we can often expect when the terms involve x^n and the series converges to a rational number for some values of x. But what led us to this easy solution was that we knew that because for various values of x the series converges to a rational number there must be a hidden cancellation pattern.

Acknowledgment

The statement of Theorem 2 originally stated that $f(n)$ must be broken into partial fractions $f_1(n), \ldots, f_j(n)$ such that $f_1(n+i_1) + \cdots + f_j(n+i_j) = 0$ for some integers $i_1, ..., i_j$. Noam D. Elkies, who was one of the first mathematicians I showed this to (around 1990) came up with the brilliantly simple counterexample $f(n) = \frac{1}{(2n-1)^2} - \frac{3}{4n^2}$ where the infinite series converges to 0. The theorem was modified to allow for "increasing (decreasing) relative occurrence of part."