Title:

On the convergence of certain infinite processes to rational numbers

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Remark 1: In my paper the following statement often appears: "Let $f(n)$ be a function which is rational for all integers n." By this I do not mean that the function f is a quotient of polynomials in n. What I mean if that f takes on rational values for all integers n. Thus we could have $f(n) = \frac{3^{n-7}}{n!}$ or $f(n) = n^n$ or $f(n) = \frac{3n^2-1}{n^4+2}$.

I realize that the term "rational function" generally means a quotient of polynomials. However I do not use the term in that sense anywhere in the paper. When I refer to functions which are quotients of polynomials I state that the function is a "quotient of polynomials".

Consider the function $f(n) = \frac{p(n)}{q(n)}$, where p and q are polynomials of n with rational coefficients. Then break $f(n)$ into partial functions. Partial fractions must be made by factorizing the denominator $q(n)$ into polynomials with rational coefficients such that the partial fractions cannot be further broken into partial fractions. In each partial fraction the degree of the numerator should be less than the degree of the denominator; further, the numerator should have only one term. Suppose , breaking into partial fractions we get $f(n) = \frac{1}{(2n-1)} - \frac{1}{(2n)}$. Then we can split one or more of these original partial fractions to get, say, $f(n) = \frac{1}{(2n-1)} - \frac{1}{(n)} + \frac{1}{(2n)}$. For theorem 2 it is important to note that partial fractions can be split up to get a new set of fractions, as shown in the above example. In theorem 2 it may happen that the original break up into partial fractions does not satisfy the equality mentioned in the theorem, but by using this method of "part splitting" we may get a different set of fractions that do satisfy the equality. "Part-splitting" results simply from writing a rational number as the sum of two or more rational numbers.

The term "translation" has the usual meaning. Let $f(n) = f_1(n) + \cdots + f_j(n)$. Then the term "translation" means replacing $f_i(n)$, for some $i (1 \leq i \leq j)$, with $f_i(n + c_i)$, where c_i is an integer.

Let $f(n) = f_1(n) + \cdots + f_j(n)$. Then define the term "increase relative occurrence of part" to mean replacing $f_i(n)$, for some $i(1 \leq i \leq j)$, with $(f_i(kn) + f_i(kn-1) + \cdots + f_i(kn-(k-1)))$, where k is a positive integer. To do the reverse replacement is to "decrease relative occurrence of part." Suppose we increase and/or decrease the relative occurrence of part p times. For each qth time $1 \leq q \leq p$ we increase relative occurrence of some part $f_i(n)$ we will need to counter this replacement by the function $D_q(n) = [-f_i(n+1) - f_i(n+2) - \cdots - f_i(kn)]$. Note that $D_q(n) = \sum_{r=1}^{n} [f_i(r) - (f_i(kr) + f_i(kr-1) + \cdots + f_i(kr-(k-1))]$. For each qth time $1 \leq j \leq p$ we decrease relative occurrence of part we will need $D_q(n) = [f_i(n+1) + f_i(n+2) + \cdots + f_i(kn)]$. q simply keeps count of how many times we have used increasing or decreasing relative occurrence of part; because q is just an independent counter it appears only on the left side in the above equation and not on the right.

Define $D(n) = D_1(n) + D_2(n) + \cdots + D_p(n)$, where each D_q is gotten by one of the above two formulas. Examples: $f(n) = \frac{1}{n} + \frac{1}{n^2}$. Increase relative occurrence of part to get $f(n) = \frac{1}{3n} + \frac{1}{3n-1} + \frac{1}{3n-2} + \frac{1}{n^2}$ and to get $D_1(n) = -\frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{3n}$. Going from the second $f(n)$ to the first would be to decrease relative occurrence of part.

In theorem 2, if on breaking $f(n)$ into partial fractions we get partial fractions of the same degree but having different coefficients associated with the powers of n, then this increased and decreased relative occurrence of part may make it possible to rearrange the partial fractions so that they satisfy the equality mentioned in theorem 2.

Theorem 2: Let $f(n)$ be a function which is either always positive or always negative for all integers $n > N$, where N is a positive integer. Further let $f(n) = \frac{p(n)}{q(n)}$, where p and q are polynomials of n with rational coefficients and let $\sum_{n=1}^{\infty} f(n)$ be convergent series. Break $f(n)$ into partial fractions, to get $f(n) = u_1(n) + \cdots + u_q(n)$. Rearrange these partial fractions by increasing or decreasing the relative occurrence of parts and by part-splitting and by translation to get a new set of fractions $f_1(n), \ldots, f_j(n)$. Then if $f_1(n) + \cdots + f_j(n) = 0$ then the infinite series is rational or irrational depending on whether $\lim_{n\to\infty} D(n)$ is rational or irrational.

If we cannot obtain $f_1(n), \ldots, f_j(n)$ such that $f_1(n) + \cdots + f_j(n) = 0$ then the infinite series is irrational.

Note that we can rearrange by using the above three methods in any order and as many times as needed.

Examples for Theorem 2:

We wish to establish whether the convergent series $\sum_{n=1}^{\infty} f(n)$ is rational or irrational for the below values of $f(n)$.

It can be seen that, if by using the three methods in any order and as many times as needed we can get $f_1(n), \ldots, f_j(n)$ such that $f_1(n) + \cdots + f_j(n) = 0$, then we can also obtain the same equality by the following procedure: first use only part-splitting, and then use only translation. The new set of fractions obtained should be such that we can just use increasing relative occurrence of parts to achieve the equality (and given a set of fractions it can be quickly seen if it is possible to rearrange them by only increasing relative occurrence of part so as to obtain the required equality).

So we can just use the above procedure and thus have an easy method of checking.

Also note that to obtain $f_1(n), \ldots, f_j(n)$ translation *alone* will generally do for series connected with important functions, and it is rare that part-splitting and increasing or decreasing relative occurrence of part will even come into play.

$f(n) = \frac{1}{(1+6n)(5+6n)}$. Taking partial fractions we get $f(n) = u_1(n) + u_2(n)$, where $u_1(n) = \frac{1}{4(1+6n)}$ and $u_2(n) = \frac{-1}{4(5+6n)}$. There exist no integers i_1, i_2 such that $u_1(n+i_1) + u_2(n+i_2) = 0$. Using part-splitting and then translation will not enable us to get a set of fractions which will satisfy the required equality by increasing relative occurrence of parts; this is because in both denominators n has the same coefficient of 6. So series irrational.

$f(n) = \frac{1}{(1+6n)(7+6n)}$. Taking partial fractions we get $f(n) = u_1(n) + u_2(n)$, where $u_1(n) = \frac{1}{6(1+6n)}$ and $u_2(n) = \frac{-1}{6(7+6n)}$. $u_1(n+1) + u_2(n+0) = 0$. So series rational.

$f(n) = \frac{1}{(2n+3)(2n+1)(2n-1)}$. Taking partial fractions we get $f(n) = u_1(n) + u_2(n) + u_3(n)$, where $u_1(n) = \frac{1}{8(2n+3)}, u_2(n) = \frac{-1}{4(2n+1)}$ and $u_3(n) = \frac{1}{8(2n-1)}$. $u_1(n+0) + u_2(n+1) + u_3(n+2) = 0$. So series rational.

$f(n) = \frac{1}{(2n+1)(2n)(2n-1)}$. Taking partial fractions we get $f(n) = u_1(n) + u_2(n) + u_3(n)$, where $u_1(n) = \frac{1}{2(2n+1)}, u_2(n) = \frac{-1}{(2n)}$ and $u_3(n) = \frac{1}{2(2n-1)}$. There exist no integers i_1, i_2, i_3 such that $u_1(n+i_1) + u_2(n+i_2) + u_3(n+i_3) = 0$. We could try part-splitting using $\frac{-1}{(2n)} = \frac{-1}{(n)} + \frac{-1}{(2n)}$. And then increase relative occurrence of $\frac{-1}{(n)}$ by replacing it with $\frac{-1}{(2n)} + \frac{-1}{(2n-1)}$. We could try this with the other partial fractions too. However, it can be seen that using part-splitting and then translation will not enable us to get a set of fractions which will satisfy the required equality by increasing relative occurrence of parts. So series irrational.

$f(n) = \frac{1}{n^k}, k$ is an integer and $k > 1$. Can't be broken into partial fractions. Using part-splitting and then translation will not enable us to get a set of fractions which will satisfy the required equality by increasing relative occurrence of parts. So series irrational.

$f(n) = \frac{(-1)^n}{n}$. The theorem applies only to series with all terms positive or negative and does not apply to alternating series. Noting that the series starts at $n = 1$ and pairing every two terms of the alternating series we get a new series equal to the alternating series. New series is $g(n) = \frac{-1}{2n-1} + \frac{1}{2n}$. We get $g(n) = u_1(n) + u_2(n)$, where $u_1(n) = \frac{-1}{2n-1}$ and $u_2(n) = \frac{1}{2n}$. There exist no integers i_1, i_2 such that $u_1(n+i_1)+u_2(n+i_2) = 0$.

However, by part-splitting we get a new set of fractions: $u_1(n) = \frac{1}{n}, u_2(n) = \frac{-1}{2n}$ and $u_3(n) = \frac{-1}{2n-1}$. Increasing the relative occurrence of $\frac{1}{n}$ we replace it by $\frac{1}{2n} + \frac{1}{2n-1}$. We get $u_1(n) = \frac{-1}{2n-1}, u_2(n) = \frac{-1}{2n}, u_3(n) = \frac{1}{2n}$ and $u_4(n) = \frac{1}{2n-1}$.$u_1(n+0) + u_2(n+0) + u_3(n+0) + u_4(n+0) = 0$. We have $D(n) = [-\frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n}]$. $\lim_{n\to\infty} D(n)$ is known to be irrational. So series irrational.

$f(n) = \frac{3}{4n^2} - \frac{1}{(2n+3)^2}$. We get $f(n) = u_1(n) + u_2(n)$, where $u_1(n) = \frac{3}{4n^2}$ and $u_2(n) = \frac{-1}{(2n+3)^2}$. There exist no integers i_1, i_2 such that $u_1(n+i_1) + u_2(n+i_2) = 0$. However, by part-splitting we get a new set of fractions: $u_1(n) = \frac{1}{n^2}, u_2(n) = \frac{-1}{(2n)^2}$ and $u_3(n) = \frac{-1}{(2n+3)^2}$. Increasing the relative occurrence of $\frac{1}{n^2}$ we replace it by $\frac{1}{(2n)^2} + \frac{1}{(2n-1)^2}$. We get $u_1(n) = \frac{1}{(2n)^2}, u_2(n) = \frac{1}{(2n-1)^2}, u_3(n) = \frac{-1}{(2n)^2}$ and $u_4(n) = \frac{1}{(2n+3)^2}$.$u_1(n+0) + u_2(n+0) + u_3(n+0) + u_4(n-2) = 0$. We have $D(n) = [-\frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} - \cdots - \frac{1}{(2n)^2}]$. $\lim_{n\to\infty} D(n)$ can be seen to converge to the rational number 0. So series rational.

$f(n) = \frac{x^n}{n(n+1)}$. Taking partial fractions we get $f(n) = u_1(n) + u_2(n)$, where $u_1(n) = \frac{x^n}{n}$ and $u_2(n) = \frac{-(x^n)}{n+1}$. If $x = 0$ or $x = 1$ then $u_1(n+1) + u_2(n+0) = 0$ and series is rational. For other positive values of x there exist no integers i_1, i_2 such that $u_1(n+i_1) + u_2(n+i_2) = 0$. Using part-splitting and then translation will not enable us to get a set of fractions which will satisfy the required equality by increasing relative occurrence of parts. So series irrational. For negative x we get an alternating series and will have to form a new series, as in previous example, by pairing terms of alternating series. We find that the series is irrational for negative x.

Actually, Theorem 2 cannot decide the rationality/irrationality of the previous series because the numerator, x^n, is not a polynomial in n. However, the stated theorem is a special case of a more general theorem — theorem 4 below.

The theorem can also be used to establish whether a number is transcendental if we know a pattern for the infinite series representation of various powers of the number. If x is algebraic then $a_0x^k + a_1x^{k-1} + ... + a_{k-1}x = a_k$, where $a_0, \ldots, a_k$ are rational. We substitute the infinite series for the powers of x in the above equation and then add the first term of each series to get a single infinite series. On this series we use the theorem to determine whether it satisfies the equation and converges to a rational number a_k or to some irrational number.

Using this method we can establish that π is transcendental by noting the following pattern about the powers of π:

for some rational constant $c_{2j}, \sum_{n=1}^{\infty} \frac{c_{2j}}{n^{2j}} = \pi^{2j}, j$ is a positive integer for some rational constant c_{2j-1}, $\sum_{n=1}^{\infty}(\frac{c_{2j-1}}{(4n-3)^{2j-1}} - \frac{c_{2j-1}}{(4n-1)^{2j-1}}) = \pi^{2j-1}$, j is a positive integer.

Remark 3: After I discovered theorem 2, I spent a considerable amount of time searching journals for proofs of irrationality and transcendence to see which *criterion* for rationality I might be able to use and how I would use it to prove theorem 2. But no proof for theorem 2 followed along known lines. Finally, it became reasonably clear that none of the criterion for rationality/irrationality were of the nature required to prove theorem 2. But that meant, "logically", that Theorem 2 can't be proved — how can the theorem be proved without using some criterion for rationality/irrationality and showing that it is violated or satisfied? The theorem can be proved by realizing that, without using any criterion for rationality, we could get a set of conditions that infinite series converging to rational numbers would satisfy but those converging to irrational numbers would not. The main difference between infinite series going to rational and irrational numbers is conditions (4) and (5) of remark 7. This change in approach immediately gives an extraordinarily elegant way to decide which real numbers are rational and which are irrational. All proofs of rationality/irrationality, for example Apery's proof of the irrationality of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ [1] — all these proofs have one thing in common: they always use some *criterion* for rationality or irrationality and show that it is violated or satisfied. It is because of this narrow approach that mathematicians have had limited success in deciding the rationality and irrationality of real numbers.

Theorem 4: Let $f(n)$ be a function which is rational for all positive integers n and which is either always positive or always negative for all integers $n > N$, where N is a positive integer. Then the series $\sum_{n=1}^{\infty} f(n)$ converges to a rational number if and only if there exist functions $f_1(n), \ldots, f_j(n)$ with $f(n) = f_1(n) + \cdots + f_j(n)$ such that

(1)$f_1(n), \ldots, f_j(n)$ are rational for all positive integers n.

(2)$f_1(n+i_1) + \cdots + f_j(n+i_j) = 0$ for some integers $i_1, \ldots, i_j$ and

(3) $\lim_{n\to\infty} \sum_{p=1}^{j}[\sum_{q=0}^{i_p-1} f_p(n+q)] = 0$.

Lemma 5: For some function $f(n)$ which is rational for all positive integers n and which is either always positive or always negative for all integers $n > N$, where N is a positive integer consider the following:

(1) There exist functions $f_1(n),\dots,f_j(n)$,which are rational for all positive integers n, such that $f(n) = f_1(n) + \cdots + f_j(n)$ and $f_1(n+i_1) + \cdots + f_j(n+i_j) = 0$ for some integers $i_1,\dots,i_j$ and $\lim_{n\to\infty}\sum_{p=1}^{j}[\sum_{q=0}^{i_p-1} f_p(n+q)] = 0$.

(2) There exists a function $F(n)$ which is rational for all positive integers n, such that $f(n) = F(n) - F(n+1)$ and $\lim_{n\to\infty} F(n) = 0$.

Then (1) is true if and only if (2) is true.

Proof of Lemma 5**:** Clearly, if (2) is true then (1) is true.

Consider (1). If $\min\{i_1,\dots,i_j\} < 0$ then add $-\min\{i_1,\dots,i_j\}$ to each integer in the set $i_1,\dots,i_j$. Call the new set of integers $i_1,\dots,i_j$.

Let $F(n) = \sum_{p=1}^{j}[\sum_{q=0}^{i_p-1} f_p(n+q)]$

Then if follows that $f(n) = f_1(n) + \cdots + f_j(n) = F(n) - F(n+1)$.

Thus if (1) is true then (2) is true.

From lemma 5 and theorem 4 we get the following theorem.

Theorem 6**:** Let $f(n)$ be a function which is rational for all positive integers n and which is either always positive or always negative for all integers $n > N$, where N is a positive integer. Then the series $\sum_{n=1}^{\infty} f(n)$ converges to a rational number if and only if there exists a function $F(n)$, such that

(1)$F(n)$ is rational for all positive integers n.

(2)$f(n) = F(n) - F(n+1)$ and

(3) $\lim_{n\to\infty} F(n) = 0$

Proof of Theorem 4 **and Theorem** 6**:** First consider theorem 6.

If $f(n) = F(n) - F(n+1)$ and $\lim_{n\to\infty} F(n) = 0$ then clearly the series converges to a rational number.

Let $F(n)$ be a function which is rational and positive for all positive integers. Let S be some positive rational number. If $S - \sum_{n=1}^{\infty} f(n) = 0$ then the first term $f(1)$ should be less than S. Say, it is less by amount $F(2)$, where $F(n)$ is a function defined on positive integers. We have f(1) = $S - F(2)$. The second term should be less than $F(2)$. Say, it is less by amount $F(3)$. Then $f(2) = F(2) - F(3)$. The n^{th} term will be $f(n) = F(n) - F(n+1)$. Since the series converges to S , the sequence $F(n)$ must also satisfy

$\lim_{n\to\infty} F(n) = 0$.

The proof is similar for negative S.

From the proof of theorem 6 and lemma 5 we see that the statement of theorem 4 is true. But theorem 4 and theorem 6 would be *unusable* unless you attach two further conditions discussed in the following Remark.

Remark 7: We know that almost all real numbers are irrational. (Here, the term "almost all" means that all reals that are not irrational have measure zero).

Trying to understand why almost all infinite series with real number terms and almost all infinite series with rational number terms would be irrational caused me to realize the new method of theorem 4/theorem 6.

Part 7 − 1:

Let us define an arithmetic function to mean a mathematical expression in variable n which involves the operators "+", "−", "×", "/" (division), and "**" (power) and uses these operators a finite number of times. It is not necessary for us to define arithmetic functions strictly in this manner. You can define an arithmetic function $f(n)$ differently, but it should be a defined as a *mathematical expression* in n, and not a "conditional" function.

$f(n) = k$, where $k = \min\{3 - f(n-1), \frac{1}{n^2}\}$ OR $f(n) = 3$ for n odd, $f(n) = n^2$ for n even OR other such definitions which are not direct mathematical expressions are not considered arithmetic functions. In the first paragraph of this paper are three examples of valid arithmetic functions.

If you find this definition of arithmetic functions to be ambiguous then define arithmetic functions to just mean functions of type $\frac{p(n)}{q(n)}$, where p and q are polynomials of n with rational coefficients. This limitation of the first definition will not cause any problems with the arguments that follow, because what holds for the broader definition will hold for this definition. If $f(n)$ is an arithmetic function then the below conditions can be added to theorem 6 (or added to theorem 4 if "$F(n)$" is replaced by "$f_k(n)$ for each $k, (1 \leq k \leq j)$".

(4) $F(n)$ must be expressible in a form other than an infinite series.

(5) $\lim_{n\to\infty} F(n)$ must be expressible in a form other than an infinite series.

"Expressible in a form other than an infinite series" means that when we write the expression for $F(n)$ or for $\lim_{n\to\infty} F(n)$, the expression must not contain an infinite series.

It is worth noting that conditions (4) and (5) cannot be expressed in a different way or a more "preferable" way. I have expressed them in the only way possible, and they are perfectly well-suited, as they are stated, to result in theorems such as theorem 2. There is no ambiguity in these conditions. Consider, say, condition (4). If it is clear what an infinite series is then given a $F(n)$ that satisfies conditions (1) through (3) we can select which $F(n)$ are infinite series. The ones *not* selected satisfy condition (4).

The above conditions (4) and (5) can be applied to the following examples:

$F(n) = (\frac{4}{7} - \sum_{k=1}^{n-1} \frac{1}{k(k+1)})$ is invalid because $\lim_{n\to\infty} F(n)$ contains an infinite series

However, using partial fractions cancellation, we can re-write the above function to get $F(n) = (\frac{4}{7} - 1 + \frac{1}{n})$ which is valid.

$F(n) = (\frac{1}{n^2} + \sum_{k=n}^{\infty} \frac{1}{k!})$ is invalid because $F(n)$ contains an infinite series.

Taken with these conditions what do theorem 4/theorem 6 tell us? What they state is that, in all series with rational terms that converge to a rational number, $f(n)$ can be broken into parts which cancel each other out. For example in theorem 6, we get $\sum_{n=1}^{\infty} f(n) = ([F(1) - F(2)] + [F(2) - F(3)] + \cdots + [F(n) - F(n+1)] + [F(n+1) - F(n+2)] + \cdots) = F(1)$. Theorem 4 will give a cancellation pattern which involves breaking into more parts, but is similar.

Choose $F(n) = (S - \sum_{k=1}^{n-1} f(k))$, where S is some rational number. This would satisfy conditions (1) and (2) of theorem 6; but it will satisfy (3) of theorem 6 if and only if, for some rational number S, $(S - \sum_{k=1}^{\infty} f(k)) = 0$. But in theorem 6 we are using conditions $(1), (2)$ and (3) to decide whether $\sum_{k=1}^{\infty} f(k)) = S$ for some rational number S. So with this choice of $F(n)$ theorem 6 will be totally circular because we cannot check condition (3) until we know whether $\sum_{k=1}^{\infty} f(k)) = S$ for some rational number S.

Similarly, choosing $F(n) = (\sum_{k=n}^{\infty} f(k))$ would satisfy conditions (2) and (3) but in deciding condition (1) we would again go into a circular loop. The added conditions (4) and (5) serve to eliminate choices such as the last two where $F(n)$ is a circular choice; these two choices of $F(n)$ are "fake" choices because $f(n)$ is not being "actually" broken into two parts, even though we have $f(n) = F(n) - F(n+1)$. Thus we see that without the above conditions (4) and (5) theorem 4 and theorem 6 would be unusable and would just be circular statements. However, with these conditions theorem 4/theorem 6 become powerful statements which can be used to examine the question of convergence to rational or irrational numbers of any infinite series whose terms are defined by some arithmetic function which is rational for all n. To sum up, if $f(n)$ is rational for all n and the series converges to a rational number then $F(n) = (S - \sum_{k=1}^{n-1} f(k))$ or $F(n) = (\sum_{k=n}^{\infty} f(k))$ will collapse down, because of some cancellation pattern, to a $F(n)$ that satisfies all conditions (1) through (5). If this $F(n)$ remains as a infinite or finite series then this $F(n)$ would violate conditions (4) and (5), respectively. So when the series converges to a rational number, $f(n)$ must be broken into parts which form an “actual” cancellation pattern, with the conditions (4) and (5) serving as tests to insure that $F(n)$ is not circular, “fake” cancellation.

The F we choose may not conform to our definition of arithmetic function (especially if you choose the second, more restrictive definition) but still must be some sort of regular *mathematical expression* since $F(n) = (S - \sum_{k=1}^{n-1} f(k))$, and f is an arithmetic function. Conditions (4) and (5) follow logically from theorem 6/ theorem 4, and therefore are just observations of the nature of $F(n)$ that is implied by theorem 6/theorem 4. How these limitations on $F(n)$ are implied by theorem 6/theorem 4 is explained below.

Instead of conditions (4) and (5) we could say that $F(n)$ must be a “closed expression” or a “closed form”; instead of (4) and (5) we could also say that the infinite series $\sum_{n=1}^{\infty} f(n)$ is a “telescoping series”.

The observation that conditions (4) and (5) must hold is what makes theorem 4/theorem 6 a workable test of rationality. The below arguments explain why it is justified to add conditions (4) and (5) to theorem 6/theorem 4.

Part $7-2$:

Let us begin by looking at a finite series of, say, 1000 terms all of which are positive rational numbers. In order to make this finite series converge to a rational number do the terms $f(1), \ldots, f(1000)$ have to be chosen in a way such that our choice of $f(n+1)$ depends on our choice of any of the previous terms $f(1), \ldots, f(n)$. Clearly not; we can choose each term independently (i.e. without worrying about what the other terms are) and the finite series will converge to a rational number.

Now suppose we want the finite series to converge to $\frac{4}{7}$. We could choose 999 terms to be any rational number less than, say $\frac{4}{7000}$. And then choose $f(1000)$ after seeing what $\sum_{k=1}^{999} f(k)$ is. To make a finite series converging to $\frac{4}{7}$ we cannot choose *all* the terms *independently* of each other; in our above way of constructing the finite series $f(1000)$ was chosen *depending* on what the others terms added to.

Now consider an infinite series converging to $\frac{4}{7}$. We will have $[\frac{4}{7} - f(1) - f(2) - \cdots - f(N)] = [f(N+1) + f(N+2) + \cdots \text{ (sum to } \infty)]$ for any N

Now for the question of rationality of the infinite series we would only be concerned that the right side be a rational number. But we are interested not that the right side be a rational number but why it would be a rational number that equals the number on the left side. Indeed in asking this question instead of the question of convergence of the right side to a rational number — in asking this question we are inventing a new criterion that infinite series that converge to rational numbers must fulfill (as explained later the criterion would still be the same if instead of $\frac{4}{7}$ we had another rational number S). Note that we are interested

only in what property f must have so as to have the above connection between the terms upto N (i.e. the terms on left) and the terms after N, no matter what N we choose; we are not investigating questions of rationality/irrationality of the right side. The right side must have some sort of "memory" of the left side because the right side must always equal not any rational number but the particular rational number on the left side. What is built into functions $f(n)$ that gives the later terms a "memory" of the previous terms?
Let us look at the matter in a new way. Suppose someone asked you to create an infinite series $\sum_{n=1}^{\infty} f(n)$ that converges to $\frac{4}{7}$. Let us say all terms of the series are positive and $f(n)$ has a rational value for all n. One would construct it this way: Let $f(1) = \frac{4}{7} - F(2)$.
Then $f(2) = F(2) - F(3)$
and $f(3) = F(3) - F(4)$
and similarly for any $n, f(n) = F(n) - F(n+1)$
When we make the infinite series by adding up the above terms a canceling chain of F is formed. (Theorem 4/theorem 6 can be proved in a simpler way. However, I have proved it in a way so that it best brings out the point that this canceling chain of F is formed.) The point is that the function f, in an infinite series that converges to $\frac{4}{7}$, is just the difference of successive terms of a sequence F of rational numbers. And you can choose any F which is rational for all integers n and $\lim_{n\to\infty} F(n) = 0$.
Now suppose we have to construct an infinite series $\sum_{n=1}^{\infty} f(n)$ as before but this time the function f should also be an arithmetic function. The F we choose may not conform to our definition of arithmetic function but, as discussed in part $7-1$, must still be some sort of regular *mathematical expression*. Thus to build such a infinite series with rational terms converging to $\frac{4}{7}$ we must choose some mathematical expression F so that $F(1) = \frac{4}{7}$ and $F(n)$ is rational for all n and $\lim_{n\to\infty} F(n) = 0$. From this chosen F we get f by putting $f(n) = F(n) - F(n+1)$. Now to say that our choice for F so as to build the required series is $F(n) = (S - \sum_{k=1}^{n-1} f(k))$ or is $F(n) = (\sum_{k=n}^{\infty} f(k))$ is a circular and logically invalid choice; (even though it is true the equalities $F(n) = (S - \sum_{k=1}^{n-1} f(k))$ and $F(n) = (\sum_{k=n}^{\infty} f(k))$ will hold once we have chosen suitable F that give us the required f that form the infinite series). The choice is invalid because the arithmetic expression f is obtained from some function F by putting $f(n) = F(n) - F(n+1)$. Now to go back in a circle and define F in terms of f is nonsense. Well, we could make F some infinite series which is known to converge to a rational number (by some other method) and say that using this F we can make an infinite series where $f(n) = F(n) - F(n+1)$. The circularity here is that F would be $F(n) = (\sum_{k=n}^{\infty} f(k))$ in order to give $f(n) = F(n) - F(n+1)$. And this this just the same situation — defining F in terms of f. Also the claim that we can get around the requirements of this method by choosing F as an infinite series that is known to converge to a rational number "by some other method" — this claim is nonsense because the

method of this paper applies to all infinite series with rational terms — if this infinite series converges to a rational number then it must satisfy conditions (1) through (5). It should thus be clear that for infinite series, with rational terms which are arithmetic functions, converging to $\frac{4}{7}$ conditions (4) and (5) can be added.

But we are not interested in conditions for an infinite series converging to the fixed rational number $\frac{4}{7}$, but converging to *any* rational number. As we have seen, for series with a finite number of terms we get a totally different situation when the finite series converges to $\frac{4}{7}$ and when it converges to any rational number.

But we never used the value to be specifically $\frac{4}{7}$. If we had any rational number S the above reasoning and conclusions would still hold.

It is interesting, however, to note that if the infinite series converges to an irrational number then the investigation of how the left side of the equation equals the right side stops as soon as we *define* some irrational number to be the sum of the infinite series. We can do this since irrational numbers can be defined and brought into existence by being designated to be sums of certain series. Suppose we have some infinite series. We can investigate (in the way of remark $7-2$) how it could go to $\frac{4}{7}$. Suppose it does not go to a rational number (but converges). Then we can just say that let us call the irrational number it goes to by some name, say "*qi*". If we make such an assignment then it makes no sense to have an investigation of what special properties the terms of the series must have so that to go to the number "*qi*". So we cannot even begin to ask the questions we asked when the series went to the rational number $\frac{4}{7}$. This is because we would be asking why it goes to "*qi*" and not to some other name. Also, it does not matter if we had an irrational number like π which also has other properties. In that case we would have an irrational number that satisfies some properties and also it can be shown to be represented by some infinite series. We call this irrational number by the name "π"; and then the same logic holds. Arguing that the name "π" came first and it was later that mathematicians found an infinite series that equalled it does not change anything. We can get statements equivalent to conditions (1) through (3) (theorem 4/theorem 6) for infinite series going to irrational numbers, but not conditions (4) and (5). And this difference follows in so subtle a way that we really use no criterion for rationality. (Of course there can also be *some* series going to irrational numbers that give closed forms. But not all infinite series converging to irrational numbers will give closed forms.)
Any real number which is not rational is *just a name given to the sum of a convergent series* – they are all just names. Consider that "square root of 2" is just a name of the convergent infinite series which has the property that its square is 2 – one could alternatively call it the "son of 2." Of course one could call $\frac{4}{7}$ the"son of $\frac{16}{49}$" but it will still be $\frac{4}{7}$ and we can ask why a series converges to this particular number –

because we have something more than a name.

This ties in with Georg Cantor's showing that almost all real numbers are irrational. Infinite series have to have a very special property to converge to a rational number and my statements imply that almost all infinite series converging will converge to irrational. Algebraic numbers also need a very special property, and we can see that from the earlier note on π being transcendental. So this is an independent way to come to Cantor's realizations that these numbers have measure zero.

I am a great admirer of Cantor (of course, his ideas are now universally acknowledged to be revolutionary) and it was very tragic that Leopold Kronecker blocked and dismissed Cantor, saying his work was simply "not mathematics." But Kronecker was, in a sense, absolutely correct when he said irrational and transcendental numbers "do not exist" and are the "work of man." *In a deeper sense* this is a central property of irrational numbers which has been exploited here to lead to new truths.

Another way to look at it is that when the infinite series goes to an irrational number then the terms can be chosen independently of each other, because as they go to infinity we have an irrational number which is a non-terminating, non-recurring decimal and so this number can just be defined to be whatever the terms add to — as we add more terms we get more decimals of this non-terminating decimal and this goes on forever; also the irrational number can be given a name "qi". If the infinite series goes to a rational number then the terms, which go on and on, have to be compatible with each other so as to go to a decimal which terminates or recurs.

The case of infinite series converging to an irrational number is very similar to the case of a finite series converging to a rational number. In the finite series case, the S can just be chosen to be whatever all the terms of the finite series add up to, just like the number "qi" can be chosen to be whatever the terms of the infinite series add to. It is worth noting again that for both infinite and finite series, with rational terms, converging to $\frac{4}{7}$ the $f(n)$ must have a memory of the sum of the previous terms, and you cannot choose each of the terms independently from each other. For infinite series converging to any rational number the f must still have this "memory"; however, for finite series converging to a rational number each of the terms may be independently chosen. In comparing finite series with rational terms converging to $\frac{4}{7}$ and infinite series with rational terms converging to any rational number one most note a major difference in that the infinite series requires the equality on p.10 to hold "for all N" (where N will run through all numbers, including an infinite number of primes etc) thus requiring a later (paired) cancelling term to also contain that same number. A finite series with rational terms converging to $\frac{4}{7}$ can go to this rational number by "coincidence" (for example consider the case of a series with only one term) and for finite series we cannot get theorems such as the ones we got for infinite series.

We said that conditions (4) and (5) hold if $f(n)$ is an arithmetic function, thus implying that they may not hold for other f. The reason for this becomes clear by considering the following example of making a series that converges to a rational number S:
Choose $f(1)$ to be any rational number between 0 and S.
Choose $f(2)$ to be a rational number between 0 and $S - f(1)$.
Choose $f(n)$ to be a rational number between 0 and $S - \sum_{k=1}^{n-1} f(k))$ and that also satisfies $S - \sum_{k=1}^{n} f(k) = 0$ as $n \to \infty$.
In this example the terms are such because each successive term $f(n)$ is chosen after considering what $S - \sum_{k=1}^{n-1} f(k))$ is.
However, if the terms are defined by some arithmetic function then the value of each term is fixed by the expression that defines $f(n)$ and cannot be chosen depending on other terms.

If my arguments above are unsatisfactory (perhaps because they do not look like mathematical proofs one generally reads), I hope they will provide the basic ideas using which you can make your own arguments to understand why conditions (4) and (5) follow. Conditions (4) and (5) follow logically – I can only explain the logic of why they follow. If you spend some time thinking about it the logic should become obvious. They may be subtle and but the logic is solid.

The reason for these arguments being the way they is is because equations, substitutions etc. which make up a normal mathematics proof won't do to prove something so general and basic. However, if you think that conditions (4) and (5) need not be true then one way you can *prove* them wrong is by finding a counter-example to theorem 2 or theorem 8, since these theorems are proved by using conditions (4) and (5). The above is a deep explanation and has to be thought about. In any case, we have to live with the fact that conditions (4) and (5) do hold in real life and I have realized that the only way to prove them is to come back to the arguments in part $7-2$. Trying totally different ways will be futile.

Part $7-3$**:**
Breaking into parts means writing a function as the sum of other functions; these other functions are called *parts* of the original function.

We say that a pair of functions *cancel* each other if for each of the pair of infinite series with these functions as terms, the additive inverse of all the terms after the Nth term is contained in the other series. (N is some positive integer)

Examples:

$\frac{1}{n}$ and $\frac{-1}{n-4}$ cancel each other by forming the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{-1}{n-4}$; the additive inverse of all the terms after $n = 5$ in the second series is in the first and the additive inverse of all the terms after $n = 5$ in the first series is in the second. In canceling convergence of individual series is not an issue.

$\frac{-1}{n}$ and $(\frac{1}{2n} + \frac{1}{2n-1})$ cancel each other by forming the series $\sum_{n=1}^{\infty} \frac{-1}{n}$ and $\sum_{n=1}^{\infty}(\frac{1}{2n} + \frac{1}{2n-1})$; even though in the first series we have alternating odd and even terms and in the second series we have both odd and even in each term. So the terms occur faster in the second series. But still the series cancel because the same fractions with opposite signs occur in both. The additive inverse of each term after $n = 1$ of the second series occurs in later terms of the first and the additive inverse of each term after $n = 1$ of the first series occurs in earlier terms of the second.

$(\frac{-1}{2n} + \frac{-1}{2n-1})$ and $(\frac{1}{3n+1} + \frac{1}{3n+2} + \frac{1}{3n+3})$ cancel each other, taking $n = 3$.

$\frac{-1}{n^2}$ and $(\frac{1}{n^2+1} + \frac{1}{n^2+2} \cdots + \frac{1}{(n+1)^2-1} + \frac{1}{(n+1)^2})$ cancel each other taking $n = 2$.

For a function g such that, for all positive integers n, $g(n)$ is a positive integer and $g(n + 1) > g(n)$ we will have $f(n)$ and $[-f(g(n) + 1) - f(g(n) + 2) - \cdots - f(g(n + 1) - 2) - f(g(n + 1) - 1) - f(g(n + 1))]$ cancel each other.

From condition (2) of theorem 4/theorem 6 it follows that if the f cannot be broken into parts such that these parts cancel each other then the series would be irrational. If it can be so broken then it would be rational or irrational depending on whether the other conditions are satisfied.

This is the procedure we will adopt to check whether for a function $f(n)$ there exist some $F(n)$ which satisfies the conditions of theorem 6:

Choose functions $g_1, g_2, \ldots, g_k$ satisfying: for $1 \leq i \leq k$, for all positive integers $n, g_i(n)$ is a positive integer and $g_i(n + 1) > g_i(n)$.

Note that k may be a positive integer or a function of n, $k(n)$, such that $k(n)$ is a positive integer for all n. To prevent the below expressions from looking too congested we will write k instead of $k(n)$; but it should be noted that for some *uncommon* kind of functions k may not be a positive integer but some other function of n.

Add: $[f(g_1(n)+1)+f(g_1(n)+2)+\cdots+f(g_1(n+1)-2)+f(g_1(n+1)-1)+f(g_1(n+1))]+[f(g_2(n)+1)+f(g_2(n)+2)+\cdots+f(g_2(n+1)-2)+f(g_2(n+1)-1)+f(g_2(n+1))]+\cdots+[f(g_k(n)+1)+f(g_k(n)+2)+\cdots+f(g_k(n+1)-2)+f(g_k(n+1)-1)+f(g_k(n+1))]$. Examine these functions to see if there are mathematical expressions in each which are additive inverses and they eliminate each other when these functions are added (for example

$\frac{1}{n^2}$ and $\frac{-1}{n^2}$). We are looking for $[f_1(g_1(n)+1)+f_1(g_1(n)+2)+\cdots+f_1(g_1(n+1)-2)+f_1(g_1(n+1)-1)+f_1(g_1(n+1))]$, a part of $[f(g_1(n)+1)+f(g_1(n)+2)+\cdots+f(g_1(n+1)-2)+f(g_1(n+1)-1)+f(g_1(n+1))]$ and $[f_2(g_2(n)+1)+f_2(g_2(n)+2)+\cdots+f_2(g_2(n+1)-2)+f_2(g_2(n+1)-1)+f_2(g_2(n+1))]$, a part of $[f(g_2(n)+1)+f(g_2(n)+2)+\cdots+f(g_2(n+1)-2)+f(g_2(n+1)-1)+f(g_2(n+1))]$ and $\cdots$ and $[f_k(g_k(n)+1)+f_k(g_k(n)+2)+\cdots+f_k(g_k(n+1)-2)+f_k(g_k(n+1)-1)+f_k(g_k(n+1))]$, a part of $[f(g_k(n)+1)+f(g_k(n)+2)+\cdots+f(g_k(n+1)-2)$ such that $[f_1(g_1(n)+1)+f_1(g_1(n)+2)+\cdots+f_1(g_1(n+1)-2)+f_1(g_1(n+1)-1)+f_1(g_1(n+1))]+[f_2(g_2(n)+1)+f_2(g_2(n)+2)+\cdots+f_2(g_2(n+1)-2)+f_2(g_2(n+1)-1)+f_2(g_2(n+1))]+\cdots+[f_k(g_k(n)+1)+f_k(g_k(n)+2)+\cdots+f_k(g_k(n+1)-2)+f_k(g_k(n+1)-1)+f_k(g_k(n+1))]=0$,
If such $f_1, f_2, \ldots, f_k$ exist then it follows that $f_1(n), f_2(n), \ldots, f_k(n)$, parts of $f(n)$, cancel each other.
Let $f'(n)=f(n)-f_1(n)-f_2(n)-\cdots-f_k(n)$.
Repeat above procedure on $f'(n)$ to find canceling parts $f'_1, f'_2, \ldots, f'_k$ to get $f''(n)=f'(n)-f'_1(n)-f'_2(n)-\cdots-f'_k(n)$. Repeat, by choosing various $g_1(n), g_2(n), \ldots, g_k(n)$ until $f'^{\cdots\prime}(n)=0$.
If a cancellation does not exist such that $f'^{\cdots\prime}(n)=0$ then series irrational. If we are able to get $f'^{\cdots\prime}(n)=0$ then form $F(n)$. To get $f'^{\cdots\prime}(n)=0$ we have been replacing $f_i(n)$ by $[f_i(g_i(n)+1)+f_i(g_i(n)+2)+\cdots+f_i(g_i(n+1)-2)+f_i(g_i(n+1)-1)+f_i(g_i(n+1))]$. To balance this replacement we need to add $f_i(n)+[-f_i(g_i(n)+1)-f_i(g_i(n)+2)-\cdots-f_i(g_i(n+1)-2)-f_i(g_i(n+1)-1)-f_i(g_i(n+1))]$ to our original series. Thus even though $f'^{\cdots\prime}(n)=0$ will result in a series with term 0 converging to the rational number 0, we will have formed a new infinite series with the functions needed to counter our replacements — we must check whether this series is rational. For this series $f_i(n)+[-f_i(g_i(n)+1)-f_i(g_i(n)+2)-\cdots-f_i(g_i(n+1)-2)-f_i(g_i(n+1)-1)-f_i(g_i(n+1))]=F_i(n)-F_i(n+1)$ will be satisfied by :
$F_i(n)=[-f_i(g_i(n)+1)-f_i(g_i(n)+2)-\cdots-f_i(n-2)-f_i(n-1)]$ when $g_i(n)<n-1$ for all positive integers $n>N$, where N is a positive integer and by
$F_i(n)=[f_i(n)+f_i(n+1)+\cdots+f_i(g_i(n)-1)+f_i(g_i(n))]$ when $g_i(n)>n-1$ for positive integers $n>N$, where N is a positive integer.
For each replacement we will get $F_i(n)$. Add all these $F_i(n)$ to get $F(n)$. This $F(n)$ must also satisfy conditions (1),(4) and (5). Then the original series will be rational or irrational number depending on condition (3) i.e if $\lim_{n\to\infty} F(n)$ is a rational or irrational number. (Note that if $\lim_{n\to\infty} F(n)$ is a rational number then we can adjust $F(n)$ by adding the additive inverse of this rational number to $F(n)$ resulting in $\lim_{n\to\infty} F(n)=0$.)

The problem will be how to know that we have exhausted all possibilities of $g_1, g_2, \ldots, g_k$ that may give us cancellations. However for series whose terms are simple constructions such as the series for theorem 2 we will have little difficulty doing the above procedure so as to exhaust all possible $g_1, g_2, \ldots, g_k$.

Proof of Theorem 2:

Throughout this proof let $f(n)$ be a function which is either always positive or always negative for all integers $n > N$, where N is a positive integer and $f(n) = \frac{p(n)}{q(n)}$, where p and q are polynomials of n with rational coefficients. Let us use above procedure on some examples of such $f(n)$.

$f(n) = \frac{1}{n(n+1)}$. Let $g_1(n) = n-1$ and $g_2(n) = n$. Adding, we get $[f(g_1(n)+1) + f(g_1(n)+2) + \cdots + f(g_1(n+1)-2) + f(g_1(n+1)-1) + f(g_1(n+1))] + [f(g_2(n)+1) + f(g_2(n)+2) + \cdots + f(g_2(n+1)-2) + f(g_2(n+1)-1) + f(g_2(n+1))] = [\frac{1}{n(n+1)}] + [\frac{1}{(n+1)(n+2)}] = [\frac{1}{n} + \frac{-1}{n+2}]$.

$f_1(n) = \frac{-1}{n+1}$ and $f_2(n) = \frac{1}{n}$, both parts of $f(n)$, cancel each other because $[f_1(g_1(n)+1) + f_1(g_1(n)+2) + \cdots + f_1(g_1(n+1)-2) + f_1(g_1(n+1)-1) + f_1(g_1(n+1))]$ and $[f_2(g_2(n)+1) + f_2(g_2(n)+2) + \cdots + f_2(g_2(n+1)-2) + f_2(g_2(n+1)-1) + f_2(g_2(n+1))]$ are additive inverses.

$f'(n) = f(n) - f_1(n) - f_2(n) = 0$ so the cancellation is complete.

$F(n) = \frac{1}{n}$ satisfies all conditions (1) through (5). So series converges to a rational number.

$f(n) = \frac{1}{2n} + \frac{-1}{2n-1}$. We have denominators n and $2n-1$ with opposite signs in numerator. This leads us to try $g_1(n) = n-1$ and $g_2(n) = 2n-2$. Adding, we get $[f(g_1(n)+1) + f(g_1(n)+2) + \cdots + f(g_1(n+1)-2) + f(g_1(n+1)-1) + f(g_1(n+1))] + [f(g_2(n)+1) + f(g_2(n)+2) + \cdots + f(g_2(n+1)-2) + f(g_2(n+1)-1) + f(g_2(n+1))] = [\frac{1}{2n} + \frac{-1}{2n-1}] + [\frac{1}{4n} + \frac{1}{4n-2} + \frac{-1}{4n-1} + \frac{-1}{4n-3}]$.

We notice we could have gotten an additive inverse pair from $\frac{-1}{2n-1}$ and $\frac{1}{4n-2}$ if the latter denominator were halved.

We try again with $f(n) = \frac{1}{n} + \frac{-1}{2n} + \frac{-1}{2n-1}$. Let $g_2(n) = 2n-2$. Adding, we get $[f(g_1(n)+1) + f(g_1(n)+2) + \cdots + f(g_1(n+1)-2) + f(g_1(n+1)-1) + f(g_1(n+1))] + [f(g_2(n)+1) + f(g_2(n)+2) + \cdots + f(g_2(n+1)-2) + f(g_2(n+1)-1) + f(g_2(n+1))] = \frac{1}{n} + \frac{-1}{2n} + \frac{-1}{2n-1} + [\frac{1}{2n} + \frac{1}{2n-1} + \frac{-1}{4n} + \frac{-1}{4n-2} + \frac{-1}{4n-1} + \frac{-1}{4n-3}]$

$f_1(n) = (\frac{-1}{2n} + \frac{-1}{2n-1})$ and $f_2(n) = \frac{1}{n}$, both parts of $f(n)$, cancel each other because $[f_1(g_1(n)+1) + f_1(g_1(n)+2) + \cdots + f_1(g_1(n+1)-2) + f_1(g_1(n+1)-1) + f_1(g_1(n+1))]$ and $[f_2(g_2(n)+1) + f_2(g_2(n)+2) + \cdots + f_2(g_2(n+1)-2) + f_2(g_2(n+1)-1) + f_2(g_2(n+1))]$ are additive inverses.

$f'(n) = f(n) - f_1(n) - f_2(n) = 0$ so the cancellation is complete.

$F(n) = (\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-3} + \frac{1}{2n-2})$ satisfies all conditions $(1), (2), (4)$, and (5). Checking condition (3) we see that $\lim_{n\to\infty} F(n)$ is irrational. So series converges to a irrational number.

$f(n) = \frac{1}{n^5}$. Adding, we get $[f(g_1(n)+1) + f(g_1(n)+2) + \cdots + f(g_1(n+1)-2) + f(g_1(n+1)-1) + f(g_1(n+1))] + [f(g_2(n)+1) + f(g_2(n)+2) + \cdots + f(g_2(n+1)-2) + f(g_2(n+1)-1) + f(g_2(n+1))] + \cdots + [f(g_k(n)+1) + f(g_k(n)+2) + \cdots + f(g_k(n+1)-2) + f(g_k(n+1)-1) + f(g_k(n+1))]$. Suppose we get some parts of each that are additive inverses. But we really need not just parts that are additive inverses of each other but $f_1(n), f_2(n), \ldots, f_k(n)$,

parts of $f(n)$, that cancel each other by having $[f_1(g_1(n)+1)+f_1(g_1(n)+2)+\cdots+f_1(g_1(n+1)-2)+f_1(g_1(n+1)-1)+f_1(g_1(n+1))]+[f_2(g_2(n)+1)+f_2(g_2(n)+2)+\cdots+f_2(g_2(n+1)-2)+f_2(g_2(n+1)-1)+f_2(g_2(n+1))]+\cdots+[f_k(g_k(n)+1)+f_k(g_k(n)+2)+\cdots+f_k(g_k(n+1)-2)+f_k(g_k(n+1)-1)+f_k(g_k(n+1))]=0$.
For convenience let us suppose that $k=2$. Then, since $f_1(n)$ and $f_2(n)$ cancel each other, $[f_1(g_1(n)+1)+f_1(g_1(n)+2)+\cdots+f_1(g_1(n+1)-2)+f_1(g_1(n+1)-1)+f_1(g_1(n+1))]$ and $[f_2(g_2(n)+1)+f_2(g_2(n)+2)+\cdots+f_2(g_2(n+1)-2)+f_2(g_2(n+1)-1)+f_2(g_2(n+1))]$ must be additive inverses.
$g_1(n)+1,\ldots,g_1(n+1)$ is a sequence of successive positive integers and $g_2(n)+1,\ldots,g_2(n+1)$ is another sequence of successive positive integers. And since for n from 1 to ∞ each of the two sequences $g_1(n)+1,\ldots,g_1(n+1)$ and $g_2(n)+1,\ldots,g_2(n+1)$ cover all integers we will get cases, for some n, of a prime number occurring in one of the sequences and not occurring in the other, (unless $g_1(n)=g_2(n)$). This means that whatever mathematical expressions $g_1(n)$ and $g_2(n)$ are, $g_1(n)+1,\ldots,g_1(n+1)$ will have some expression(s) which are relatively prime to the other expressions in $g_1(n)+1,\ldots,g_1(n+1)$ and $g_2(n)+1,\ldots,g_2(n+1)$. For example if $g_1(n)=n^2$ and $g_2(n)=n^3$ then $g_1(n)+1=n^2+1$ would be relatively prime to each of $g_1(n)+2,\ldots,g_1(n+1)$ and $g_2(n)+1,\ldots,g_2(n+1)$, as they would have no factors in common. Suppose $g_1(n)+1$ is an expression that is relatively prime to all of $g_1(n)+2,\ldots,g_1(n+1)$ and $g_2(n)+1,\ldots,g_2(n+1)$. $f(g_1(n)+1)=\frac{1}{(g_1(n)+1)^5}$ has a denominator relatively prime to all other denominators. Adding, we get $[f(g_1(n)+1)+f(g_1(n)+2)+\cdots+f(g_1(n+1)-2)+f(g_1(n+1)-1)+f(g_1(n+1))]+[f(g_2(n)+1)+f(g_2(n)+2)+\cdots+f(g_2(n+1)-2)+f(g_2(n+1)-1)+f(g_2(n+1))]$. Suppose we get some parts of each that are additive inverses. But we really need not just parts that are additive inverses of each other but $f_1(n), f_2(n)$, parts of $f(n)$, that cancel each other by having $[f_1(g_1(n)+1)+f_1(g_1(n)+2)+\cdots+f_1(g_1(n+1)-2)+f_1(g_1(n+1)-1)+f_1(g_1(n+1))]$ and $[f_2(g_2(n)+1)+f_2(g_2(n)+2)+\cdots+f_2(g_2(n+1)-2)+f_2(g_2(n+1)-1)+f_2(g_2(n+1))]$ as additive inverses. But how do we get a $f_1(g_1(n)+1)$, a part of $f(g_1(n)+1)$, that is the additive inverse of parts from other terms. From the fact that $f(g_1(n)+1)=\frac{1}{(g_1(n)+1)^5}$ has a denominator relatively prime to all other denominators if follows that f_1, f_2 don't cancel each other.
If it is not clear why, I give a detailed explanation below. The explanation is given for general $f(n)$.

Explanation: The below pairs of functions are defined to be "addition compatible functions":
$\frac{1}{n(n+1)}$ and $\frac{1}{n+1}$. Note $\frac{1}{n(n+1)}=\frac{1}{n}+\frac{-1}{n+1}$.
$\frac{1}{n(n+1)}$ and $\frac{1}{2(n+1)}$
$\frac{1}{n(n+1)}$ and $\frac{1}{n}$
$\frac{1}{n(n+1)(n+2)}$ and $\frac{1}{n+1}$. Note $\frac{1}{n(n+1)(n+2)}=\frac{1}{2n}+\frac{-1}{n+1}+\frac{1}{2(n+2)}$.
So addition compatible functions are those which, on breaking into partial fractions contain at least one

common partial fraction, that differs just by a coefficient.

Any pair of functions can be broken into parts such that some of the parts from one are additive inverses of parts of the other, but not any pair of functions will are addition compatible.

The below pairs of functions are not addition compatible.

$\frac{1}{n(n+1)}$ and $\frac{1}{n+2}$

$\frac{1}{n(n+1)(n+2)}$ and $\frac{1}{n-7}$

"Addition compatible" is not a rigorous definition in its application to functions other than quotients of polynomials, but as we see later, it is not needed to be for our purpose.

Use the procedure in part $7-3$ which says:

Let $f'(n) = f(n) - f_1(n) - f_2(n) - \cdots - f_k(n)$.

Repeat above procedure on $f'(n)$ to find canceling parts $f'_1, f'_2, \ldots, f'_k$ to get $f''(n) = f'(n) - f'_1(n) - f'_2(n) - \cdots - f'_k$. Repeat, by choosing various $g_1(n), g_2(n), \ldots, g_k(n)$ until $f'^{\cdots\prime}(n) = 0$.

Note that, in part $7-3$, $f'(n), f''(n)$ etc. are used only for convenience. The following can be seen to be true: if, for some $f(n)$, following the above steps we can get $f'^{\cdots\prime}(n) = 0$ then there must exist $f_1, f_2, \ldots, f_k$ such that we get $f(n) - f_1(n) - f_2(n) - \cdots - f_k(n) = 0$, in one single step.

So in this proof, to see whether $f(n)$ can be broken into canceling parts we only worry about whether there exist $f_1(n), f_2(n), \ldots, f_k(n)$, parts of $f(n)$, that cancel each other and give $f(n) - f_1(n) - f_2(n) - \cdots - f_k(n) = 0$. If no such $f_1(n), f_2(n), \ldots, f_k(n)$ exist then the series cannot be broken into canceling parts and is therefore irrational.

For convenience, let us assume that in $f_1, \cdots, f_k$, $k = 2$.

We have $f(n) = f_1(n) + f_2(n)$, and $f_1(n)$ and $f_2(n)$ cancel each other i.e. $[f_1(g_1(n)+1) + f_1(g_1(n)+2) + \cdots + f_1(g_1(n+1)-2) + f_1(g_1(n+1)-1) + f_1(g_1(n+1))] + [f_2(g_2(n)+1) + f_2(g_2(n)+2) + \cdots + f_2(g_2(n+1)-2) + f_2(g_2(n+1)-1) + f_2(g_2(n+1))] = 0$

Let $f_1(n) = h_1(n) + h(n)$ and $f_2(n) = h_2(n) - h(n)$ such that $h_1(n)$ and $h_2(n)$ are obtained by adding together one or more partial fractions of $f(n)$ or by multiplying these partial fractions by rational constants and then adding them together. Throughout this proof we will assume that h_1 and h_2 *have to be formed from partial fractions of $f(n)$ in this described way.* Suppose f_1 and f_2 cancel each other but there exist no h_1 and h_2 such that $h(n) = 0$. We show that this cannot occur. We show that if there exist f_1 and f_2 that cancel each other (and satisfy conditions (1) through (5)) then there must exist h_1 and h_2 such that $h(n) = 0$.

Consider what we actually mean by parts canceling. Suppose $f_1(n), f_2(n), \ldots, f_k(n)$, parts of $f(n)$, cancel each other.

This means that if you make k columns and write one function in each column for all n then we can cancel out functions in each column that are additive inverses and will be left with a finite number of uncancelled terms in each column.

Example $f(n) = \frac{1}{n} - \frac{1}{n+2}$. We get two columns:

$\frac{1}{n}$	$\frac{-1}{n+2}$
$\frac{1}{n+1}$	$\frac{-1}{n+3}$
$\frac{1}{n+2}$	$\frac{-1}{n+4}$
$\frac{1}{n+3}$	$\frac{-1}{n+5}$

etc.

In the second column no terms are left uncancelled whereas in the first column the first two terms are left uncancelled.

Suppose f_1 and f_2 cancel each other but we cannot find h_1 and h_2 as described above such that $h(n) = 0$. Make columns with h_1 and h_2. By hypotheses, these two columns don't cancel each other. Also, by hypotheses, we get a canceling f_1, f_2 by adding $h(n)$ and $-h(n)$ to $h_1(n)$ and $h_2(n)$ respectively. First, suppose we choose a $h(n)$ that is obtained by adding together one or more partial fractions of $f(n)$ or by multiplying these partial fractions by rational constants and then adding them together.

Example: $h_1(n) = \frac{1}{n(n+1)}$ and $h_2(n) = 0$. Let $h(n) = \frac{1}{(n+1)}$. Then $f_1(n) = \frac{1}{n}$ and $f_2(n) = \frac{-1}{(n+1)}$. We have gotten a $f_1(n), f_2(n)$ that cancel each other.

But the problem with this example and any other example where we get $h(n)$ by adding together rational constant times the partial fractions of $f(n)$ is that *we could have chosen h_1 and h_2 differently so that we did not need such a $h(n)$ at all* i.e. we could have chosen suitable h_1, h_2 so that $h(n) = 0$ (by choosing $h_1(n) = f_1(n)$ and $h_2(n) = f_2(n)$ above).So choosing addition compatible $h(n)$ (i.e. $h(n)$ that are obtained by adding together one or more partial fractions of $f(n)$ or by multiplying these partial fractions by rational constants and then adding them together) — choosing such $h(n)$ is no use.

Now, suppose we choose a $h(n)$ that is *not of the previous type*, say we choose $h(n) = \frac{1}{(n+3)}$. Then let us write f_1 and f_2 as two columns.We get:

f_1	f_2
$\frac{1}{n(n+1)} + \frac{1}{n+3}$	$\frac{-1}{n+3}$
$\frac{1}{(n+1)(n+2)} + \frac{1}{n+4}$	$\frac{-1}{n+4}$
$\frac{1}{(n+2)(n+3)} + \frac{1}{n+5}$	$\frac{-1}{n+5}$
$\frac{1}{(n+3)(n+4)} + \frac{1}{n+6}$	$\frac{-1}{n+6}$

etc.

We notice that the f_1 column is itself made of two columns and between these columns we can get a cancellation by breaking the left column of f_1 into partial fractions and translating the right column of f_1 by -2. By this translation $\frac{1}{n+3}$ goes to $\frac{1}{n+1}$, $\frac{1}{n+4}$ goes to $\frac{1}{n+2}$ etc. After doing this we get a f_1 and f_2 column that cancel each other. But to achieve this we had to first do a cancellation *within* f_1. But we started by assuming that $[f_1(g_1(n)+1)+f_1(g_1(n)+2)+\cdots+f_1(g_1(n+1)-2)+f_1(g_1(n+1)-1)+f_1(g_1(n+1))]+[f_2(g_2(n)+1)+f_2(g_2(n)+2)+\cdots+f_2(g_2(n+1)-2)+f_2(g_2(n+1)-1)+f_2(g_2(n+1))]=0$ which does not allow any cancellations *within* f_1. Alternatively, we can see that the definition of two functions canceling (on p. 14) does not allow cancellations *within* f_1.

(Of course if we had added a h such that $h_1(n)$ and $h(n)$ were addition compatible or $h_2(n)$ and $-h(n)$ were addition compatible then we would need no internal cancellations of the above sort within f_1 and f_2 as simply adding these two functions together would be effective — as in the example on the previous page).

The reason why adding $\frac{1}{n+3}$ did not get f_1 and f_2 to cancel can be generalized to below $h(n)$.

Suppose we had chosen any $h(n)$ such that $h(n)$ and $h_1(n)$ are not addition compatible and also $-h(n)$ and $h_2(n)$ are not addition compatible then we will not be able to get f_1 and f_2 to cancel (without violating conditions (4) or (5)) — even if this $h(n)$ allowed us to get the $\frac{1}{n+3}$ example sort of cancellation *within* f_1 or f_2, as such cancellations are not allowed (as explained above), only direct addition is allowed.

Suppose we look for a function that is not addition compatible then it can be seen that trying to construct such a function leads to a choice that violates conditions (4) or (5). To see this consider the sample case $g_1(n)=n$ and $g_2(n)=n-1$.

From these g_1, g_2 we get $f_1(n+1)=-f_2(n)$. So $h_1(n+1)+h(n+1)=-h_2(n)+h(n)$. From this we get $h(n)=h_1(n+1)+h_2(n)+h(n+1)$ **(Eq. 1)**

But Eq. 1 clearly shows that such a definition of $h(n)$ will make it a finite series (from 1 to n) or an infinite series. Below we construct this series. (To see why we will get a series assume $h_2(n)=0$. Then we get $h(n+1)=h(n)-h_1(n+1)$ which shows $h(n+1)$ is gotten *each time* by adding a (negative) term to $h(n)$.)

We need a h that will satisfy Eq. 1.

From Eq. 1 we get $h(n+1) = h_1(n+2) + h_2(n+1) + h(n+2)$ **(Eq. 2)**

From Eq. 2 we get $h(n+2) = h_1(n+3) + h_2(n+2) + h(n+3)$ **(Eq. 3)**

And we can continue getting such equations.

Substituting Eq. 2 into Eq. 1, Eq. 3 into Eq. 2 and so on we get:

$h(n) = \sum_{k=n}^{\infty}(h_1(k+1) + h_2(k))$. **(Eq. 4)**

In Eq. 2, Eq. 3 etc. if we had found $h(n-1), h(n-2)$ etc. on the left side we would end up with $h(n) = \sum_{k=1}^{n-1}(-h_1(k+1) - h_2(k))$.

These choices violate conditions (4), (5). If we had started with different g_1, g_2 we will still get a violation of conditions (4) or (5), though Eq. 1 would look more complicated . But to satisfy conditions (4), (5) $h(n) = \sum_{k=n}^{\infty}(h_1(k+1) + h_2(k))$ must have functions which are addition compatible so they add up and eliminate each other leaving a $h(n)$ that satisfies condition (4), (5).

This can happen only if some of these addition compatible functions would be additive inverses — $\frac{1}{n^2}$ and $\frac{-1}{n^2}$ are examples of addition compatible function which are additive inverses and which would eliminate each other if they occurred in different terms of infinite series in Eq. 4. Suppose $\sum_{k=n}^{\infty} h_1(k+1)$ is such that some of the terms are addition compatible functions (and some of these addition compatible functions might be additive inverses). But clearly that would contradict the fact that $h_1(n)$ and $h(n)$ are not addition compatible.

(Why would it contradict? Think of it like a chain of h_1, and $h_1(n)$ would be a link in this chain — if functions from various terms are eliminating each other then for $h_1(n)$ there must be some function, in one of the h_1 terms after it, that is addition compatible with $h_1(n)$. This function will be left after additive inverses eliminate each other in $h(n)$, giving a $h(n)$ that satisfies conditions (4), (5) — but the resulting $h(n)$ will be addition compatible with $h_1(n)$.)

We get the same conclusion by looking at $\sum_{k=n}^{\infty} h_2(k)$.

So the only other possible way to get a $h(n)$ that satisfies conditions (4), (5) and also is not addition compatible with h_1, h_2 is that the terms of the infinite series formed by $h_1(n+1)$ must be eliminating terms (by having additive inverses eliminate each other on addition) of the infinite series formed by $h_2(n)$; but this means that h_1, h_2 cancel each other which is a contradiction.

Note: One can counter the above argument by saying that in $\sum_{k=n}^{\infty} h_1(k+1)$ some the terms are additive inverses (or are addition compatible functions) and these eliminate each other and what is left of the h_1 infinite series is eliminated by $h_2(n)$ etc. So then $h(n)$ and $h_1(n)$ are not addition compatible, as we concluded above they would be in such a case. But it can be seen that if this happens then it would again just imply that h_1 and h_2 cancel each other.

Examples that can be tried to understand above if the above is not clear:

$h_1(n) = \frac{1}{n(n+1)}$, $h_2(n) = 0$. The h_1 form a chain such that we get $h(n) = \frac{1}{n+1}$. But this is addition compatible with $h_1(n)$.

$h_1(n) = \frac{1}{n}$, $h_2(n) = \frac{-1}{n+2}$. We get a $h(n) = \frac{1}{n+1}$ resulting from terms in infinite series formed by $h_1(n)$ eliminating terms of the infinite series formed by $h_2(n)$. But is such cases we get from the above arguments that h_1, h_2 cancel each other. And here we can see that indeed they do without the need to add any $h(n)$.

Thus we see that choosing $h(n)$ such that $h(n)$ and $h_1(n)$ are not addition compatible and also $-h(n)$ and $h_2(n)$ are not addition compatible — choosing this *second kind* of $h(n)$ serves no purpose. Also we have seen that choosing the *first kind* of $h(n)$ i.e. choosing $h(n)$ to be the sum of rational constant times the partial fractions is no use.

Suppose we choose $h(n)$ to be the sum of the above two possible choices so that we get a "mixed" kind of $h(n)$ which contains partial fractions but also stuff that has nothing to do with partial fractions. But instead of making $h(n)$ the sum of functions of both types we can remove from $h(n)$ functions of the first kind by choosing different h_1, h_2 so that we do not need the partial fractions in $h(n)$ (as explained above). Then $h(n)$ is just a function of the second kind and such a function serves no purpose.

So if f_1 and f_2 cancel then we only have to look for suitable h_1 and h_2 and we can assume $h(n) = 0$ since taking any kind of non-zero h serves no purpose.

The key to the above proof was that it is easy for the kind of $f(n)$ we are considering to decide which $h(n)$ are addition compatible — these are the $h(n)$ obtained by summing rational constant times the partial fractions of $f(n)$.

We can make some generalizations from the above arguments.

The $f_1(n), \ldots, f_k(n)$ that cancel each other should each be a constant times the partial fractions of $f(n)$.

Consider $f(n) = f_1(n) + f_2(n)$. Suppose $f_1(n)$ and $[f_2(g_1(n)+1)f_2(g_1(n)+2) + \cdots + f_2(g_1(n+1)-2) + f_2(g_1(n+1)-1) + f_2(g_1(n+1))]$ are additive inverses. Then $f(n) = [-f_2(g_1(n)+1) - f_2(g_1(n)+2) - \cdots - f_2(g_1(n+1)-2) - f_2(g_1(n+1)-1) - f_2(g_1(n+1))] + f_2(n)$. But $f(n)$ must contain a constant number of partial fractions. This implies that $g_1(n)$ can only be polynomials of degree 1. So when using the procedure of $7-3$ we only have to try $g_1(n)$ and $g_2(n)$ which are polynomials of degree 1.

Consider $f_1(n)$ and $f_2(n)$ of form $\frac{p(n)}{q(n)}$, where p and q are polynomials of n with rational coefficients and $f(n) = [f_1(g_1(n)+1)+f_1(g_1(n)+2)+\cdots+f_1(g_1(n+1)-2)+f_1(g_1(n+1)-1)+f_1(g_1(n+1))]+[f_2(g_2(n)+1)+f_2(g_2(n)+2)+\cdots+f_2(g_2(n+1)-2)+f_2(g_2(n+1)-1)+f_2(g_2(n+1))]$ and either one or both of $g_1(n)$ and $g_2(n)$ are not polynomials of degree 1. Then if $f(n)$ is of form $\frac{p(n)}{q(n)}$, where p and q are polynomials of n with rational coefficients then there must exist $g_3(n)$ and $g_4(n)$ which are polynomials of degree 1 such that $[f_1(g_3(n)+1)+f_1(g_3(n)+2)+\cdots+f_1(g_3(n+1)-2)+f_1(g_3(n+1)-1)+f_1(g_3(n+1))]$ and $[f_2(g_4(n)+1)+f_2(g_4(n)+2)+\cdots+f_2(g_4(n+1)-2)+f_2(g_4(n+1)-1)+f_2(g_4(n+1))]$ are additive inverses. Generalizing the above statements we see that, in all cases, we only have to worry about what remark 1 describes as "increasing or decreasing relative occurrence of part".

If $f(n) = f_1(g_1(n)+1)+f_1(g_1(n)+2)+\cdots+f_1(g_1(n+1)-2)+f_1(g_1(n+1)-1)+f_1(g_1(n+1))]+[-f_1(g_2(n)+1)-f_1(g_2(n)+2)-\cdots-f_1(g_2(n+1)-2)-f_1(g_2(n+1)-1)-f_1(g_2(n+1))]$ then some of these partial fractions having the same polynomial in their denominators may combine, for example $\frac{1}{n}$ and $\frac{-1}{2n}$ (in the second example in this proof), and therefore we need part splitting in theorem 2.

Note than theorem 2 would still hold if *only* "increasing relative occurrence" rather than both "increasing and decreasing relative occurrence" were used. Also note that $D(n)$ covers only cases of increasing and decreasing relative occurrence and for these cases $D(n) = F(n+1)$. We neglect cases of translation since these do not affect condition (3).

Thus theorem 2 follows.

On the same line of reasoning as above we can get theorems about other infinite processes converging to rational numbers. Consider $f(n) = \frac{p(n)}{q(n)}$, where p and q are polynomials of n with rational coefficients. We define increased and decreased relative occurrence of parts similarly as in remark 1 but replace "$+f_i$" by "$\times f_i$" (multiplication) and replace "$-f_i$" by "$\times \frac{1}{f_i}$"

Factorizing $f(n)$ means breaking into factors such that these factors multiplied together give $f(n)$. Each of these factors must also be of type $\frac{p(n)}{q(n)}$, with either $p(n) = 1$ or $q(n) = 1$. Further $p(n)$ and $q(n)$ should be such that they cannot be further factorized. Thus $p(n) = n^2$ is not correct since n^2 can be further factorized to get $n \times n$. Part-splitting means writing a rational number as the product of two rational numbers and placing these rational numbers with any of the factors. Suppose $f(n) = \frac{3n(n+1)}{4(n-2)(n+2)}$. Then by part splitting we can get factors $f_1(n) = 3n$, $f_2(n) = (n+1)$, $f_3(n) = \frac{1}{2(n-2)}$, and $f_4(n) = \frac{1}{2(n+2)}$ OR $f_1(n) = 5n$, $f_2(n) = 3(n+1)$, $f_3(n) = \frac{1}{5(n-2)}$, and $f_4(n) = \frac{1}{4(n+2)}$ OR other possibilities by moving the numbers 3 and 4 between different factors and writing 1 as $\frac{5}{5}$ etc.

Theorem 8: Let $f(n)$ be a function which is either always positive or always negative for all integers $n > N$, where N is a positive integer. Further let $f(n) = \frac{p(n)}{q(n)}$, where p and q are polynomials of n with rational coefficients and let $\prod_{n=1}^{\infty} f(n)$ be converge to a number other than 0. Factorize $f(n)$ to get $f(n) = u_1(n) \times \cdots \times u_q(n)$. Rearrange by increasing or decreasing the relative occurrence of parts and by part-splitting and by translation to get a new set of fractions $f_1(n), \ldots, f_j(n)$. Then if $f_1(n) \times \cdots \times f_j(n) = 1$ then the infinite product is rational or irrational depending on whether $\lim_{n\to\infty} D(n)$ is rational or irrational. If we cannot obtain $f_1(n), \ldots, f_j(n)$ such that $f_1(n) \times \cdots \times f_j(n) = 1$ then the infinite product is irrational. Note that we can rearrange by using the above three methods in any order and as many times as needed.

Examples for Theorem 8:

We wish to establish whether the infinite product, $\prod_{n=1}^{\infty} f(n)$, convergent to a non-zero real number, is rational or irrational for the below values of $f(n)$.

$f(n) = 1 - \frac{1}{(2n+1)^2}$. Factorizing into parts we get $f(n) = f_1(n) \times f_2(n) \times f_3(n) \times f_4(n)$, where $f_1(n) = \frac{1}{2n+1}, f_2(n) = \frac{1}{2n+1}, f_3(n) = 2n$, and $f_4 = 2n + 2$. There exist no integers i_1, i_2, i_3, i_4 such that $f_1(n + i_1) \times f_2(n + i_2) \times f_3(n + i_3) \times f_3(n + i_3) = 1$. Increasing or decreasing relative occurrence will not allow us to get the appropriate cancellation. So infinite product is irrational.

Consider the series below for $\prod_{n=2}^{\infty} f(n)$
$f(n) = \frac{n^3-1}{n^3+1}$. Factorizing into parts we get $f(n) = f_1(n) \times f_2(n) \times f_3(n) \times f_4(n)$, where $f_1(n) = \frac{1}{n+1}, f_2(n) = \frac{1}{1-n+n^2}, f_3(n) = (n-1)$, and $f_4(n) = (1+n+n^2)$. $f_1(n) \times f_2(n+1) \times f_3(n+2) \times f_4(n+0) = 1$. So infinite product rational.

Theorem 9: Let $f(n)$ be a function which is rational for all positive integers n and which is positive for all integers $n > N$, where N is a positive integer. Further, let $\prod_{n=1}^{\infty} f(n)$ not be convergent to 0. Then the infinite product converges to a rational number if and only if there exist functions $f_1(n), \ldots, f_j(n)$ with $f(n) = f_1(n) \times \cdots \times f_j(n)$ such that

(1)$f_1(n), \ldots, f_j(n)$ are rational for all positive integers n.

(2)$f_1(n) \times \cdots \times f_j(n) = 1$ for some integers $i_1, \ldots, i_j$ and

(3) $\lim_{n\to\infty} \prod_{p=1}^{j} [\prod_{q=0}^{i_p - 1} f_p(n+q)] = 1$

Theorem 10**:** Let $f(n)$ be a function which is rational for all positive integers n and which is positive for all integers $n > N$, where N is a positive integer. Further, let $\prod_{n=1}^{\infty} f(n)$not be convergent to 0. Then the infinite product converges to a rational number if and only if there exist a function $F(n)$ such that

(1)$F(n)$ is rational for all positive integers n.

(2)$f(n) = \frac{F(n)}{F(n+1)}$ and

(3) $\lim_{n\to\infty} F(n) = 1$

Remark 11**:** I have not illustrated the widespread application of my theorem, except in the case of series and infinite products of quotients of polynomials. It is fairly easy, based on the same lines as the proof of theorem 2, to establish whether other kinds of series, such as those that involve factorials, are rational or irrational. However, there are some series that are a little more challenging, even with these theorems. For example, theorem 4 cannot easily determine whether series that involve x^n, where x is a constant will be rational or irrational; this is because replacing $x^n = (y - z)^n$, where y and z are constants, will cause new possibilities of cancellation. I shall discuss some such individual series more fully in a future paper. I chose to illustrate the use of my approach for series whose terms are quotients of polynomials because in this case we are able to form a single test for this family of infinite series. For other series a case by case application of the theorems will generally be needed. To establish whether an infinite series is rational or irrational one simply has to establish whether the terms are compatible in such a way that they will cancel each other out. This should be the logical and standard way to decide rationality and irrationality of numbers for whom the infinite series equal, having terms expressed as a function which takes rational values for all n, is known. Using this method, it is fair to say that the irrationality of infinite series, infinite products, and various other infinite processes now becomes a matter of routine labor rather than a serious intellectual challenge requiring original investigations in number theory.

We can also use the ideas of these methods to analyze infinite series, and learn facts about them other than what kind of real number they converge to. To illustrate this we solve the following problem quoted from a 1826 paper by Neils H. Abel [2]:

"One of the most remarkable series of algebraic analyses is the following: $1+\frac{m}{1}x+\frac{m(m-1)}{1\cdot 2}x^2+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3+\cdots+\frac{m(m-1)\cdots[m-(n-1)]}{1\cdot 2\cdots\cdot n}x^n+\cdots$

When m is a positive whole number the sum of the series, which is then finite, can be expressed, as is known, by $(1+x)^m$. When m is not an integer, the series goes on to infinity, and it will converge or diverge according as the quantities m and x have this or that value. In this case one writes the same equality $(1+x)^m = 1+\frac{m}{1}x+\frac{m(m-1)}{1\cdot 2}x^2+\cdots$ etc.

$\cdots$ It is assumed that the numerical equality will always occur whenever the series is convergent, but this has never yet been proved."

Now an alternate way to prove this would be that if we knew the infinite series converges to a rational number then there must exist a cancellation pattern. A search for a cancellation pattern, say for the case $m=\frac{1}{2}$ leads to replacing x by y^2-1. Also further replace $y^2-1=(y-1)(y+1)$ by $z(z-2)$, putting $z=1-y$. In each term replacing x with z^2-2z, expanding each term, and adding the same powers of z from all terms, gives us the required cancellation pattern. When $m=\frac{1}{3}$ we would begin by replacing x with y^3-1 etc. Thus this numerical equality is proved by finding and using this cancellation pattern, without actually using any of the theorems in this paper. This example is unusual because here we have a cancellation pattern for both cases — when the series converges to a rational or to an irrational number; this kind of occurrence is what we can often expect when the terms involve x^n and the series converges to a rational number for some values of x. But what led us to this easy solution was that we knew that because for various values of x the series converges to a rational number there must be a hidden cancellation pattern.

Remark 12**:** Though this paper deals with classifying real numbers into rational and irrational we can, along the same lines, develop theorems about other kinds of numbers. The basic idea that leads to the above theorems can be directly applied to infinite processes involving numbers that satisfy some property, to determine whether the number the infinite process converges to will satisfy the same property. The major deciding factor will be how the numbers are defined.

As mentioned in remark 7, because irrational numbers are defined in a *negative* manner, (every number which is *not* a quotient of integers is called irrational), we cannot have an algorithm such as Theorem 2 that will decide whether an infinite series with all irrational terms will converge to an irrational.

So this paper is as much about number theory as about abstract algebra since we never used any properties as such of rational numbers or irrational numbers. Fields constructed similar to the way rational numbers are constructed will give similar theorems. The expression of this paper into formal algebraic terms should be possible.

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1. The statement of Theorem 2 originally stated that $f(n)$ must be broken into partial fractions $f_1(n),\ldots,f_j(n)$ such that $f_1(n+i_1)+\cdots+f_j(n+i_j)=0$ for some integers $i_1,...,i_j$. One of the first mathematicians (name withheld from this preprint) I showed this to came up with the brilliantly simple counterexample $f(n)=\frac{1}{(2n-1)^2}-\frac{3}{4n^2}$ where the infinite series converges to 0. The theorem was modified to allow for "increasing (decreasing) relative occurrence of part."