Title:

On the convergence of certain infinite processes to rational numbers

(Text of paper reduced to focus on Mathematical Logic issue)

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Remark 1: In my paper the following statement often appears: "Let $f(n)$ be a function which is rational for all integers n." By this I do not mean that the function f is a quotient of polynomials in n. What I mean if that f takes on rational values for all integers n. Thus we could have $f(n) = \frac{3^{n-7}}{n!}$ or $f(n) = n^n$ or $f(n) = \frac{3n^2-1}{n^4+2}$.

I realize that the term "rational function" generally means a quotient of polynomials. However I do not use the term in that sense anywhere in the paper. When I refer to functions which are quotients of polynomials I state that the function is a "quotient of polynomials".

Theorem 4: Let $f(n)$ be a function which is rational for all positive integers n and which is either always positive or always negative for all integers $n > N$, where N is a positive integer. Then the series $\sum_{n=1}^{\infty} f(n)$ converges to a rational number if and only if there exist functions $f_1(n), \ldots, f_j(n)$ with $f(n) = f_1(n) + \cdots + f_j(n)$ such that

(1)$f_1(n), \ldots, f_j(n)$ are rational for all positive integers n.

(2)$f_1(n+i_1) + \cdots + f_j(n+i_j) = 0$ for some integers $i_1, \ldots, i_j$ and

(3) $\lim_{n\to\infty} \sum_{p=1}^{j} [\sum_{q=0}^{i_p-1} f_p(n+q)] = 0$.

Lemma 5: For some function $f(n)$ which is rational for all positive integers n and which is either always positive or always negative for all integers $n > N$, where N is a positive integer consider the following:

(1) There exist functions $f_1(n), \ldots, f_j(n)$,which are rational for all positive integers n, such that $f(n) = f_1(n) + \cdots + f_j(n)$ and $f_1(n+i_1) + \cdots + f_j(n+i_j) = 0$ for some integers $i_1, \ldots, i_j$ and $\lim_{n\to\infty} \sum_{p=1}^{j} [\sum_{q=0}^{i_p-1} f_p(n+q)] = 0$.

(2) There exists a function $F(n)$ which is rational for all positive integers n, such that $f(n) = F(n) - F(n+1)$ and $\lim_{n\to\infty} F(n) = 0$.

Then (1) is true if and only if (2) is true.

Proof of Lemma 5: Clearly, if (2) is true then (1) is true.

Consider (1). If $\min\{i_1, \ldots, i_j\} < 0$ then add $-\min\{i_1, \ldots, i_j\}$ to each integer in the set $i_1, \ldots, i_j$. Call the new set of integers $i_1, \ldots, i_j$.

Let $F(n) = \sum_{p=1}^{j} [\sum_{q=0}^{i_p-1} f_p(n+q)]$

Then if follows that $f(n) = f_1(n) + \cdots + f_j(n) = F(n) - F(n+1)$.

Thus if (1) is true then (2) is true.

From lemma 5 and theorem 4 we get the following theorem.

Theorem 6: Let $f(n)$ be a function which is rational for all positive integers n and which is either always positive or always negative for all integers $n > N$, where N is a positive integer. Then the series $\sum_{n=1}^{\infty} f(n)$ converges to a rational number if and only if there exists a function $F(n)$, such that

(1)$F(n)$ is rational for all positive integers n.

(2)$f(n) = F(n) - F(n+1)$ and

(3) $\lim_{n\to\infty} F(n) = 0$

Proof of Theorem 4 and Theorem 6: First consider theorem 6.

If $f(n) = F(n) - F(n+1)$ and $\lim_{n\to\infty} F(n) = 0$ then clearly the series converges to a rational number.

Let $F(n)$ be a function which is rational and positive for all positive integers. Let S be some positive rational number. If $S - \sum_{n=1}^{\infty} f(n) = 0$ then the first term $f(1)$ should be less than S. Say, it is less by amount $F(2)$, where $F(n)$ is a function defined on positive integers. We have f(1) = $S - F(2)$. The second term should be less than $F(2)$. Say, it is less by amount $F(3)$. Then $f(2) = F(2) - F(3)$. The n^{th} term will be $f(n) = F(n) - F(n+1)$. Since the series converges to S , the sequence $F(n)$ must also satisfy $\lim_{n\to\infty} F(n) = 0$.

The proof is similar for negative S.

From the proof of theorem 6 and lemma 5 we see that the statement of theorem 4 is true. But theorem 4 and theorem 6 would be *unusable* unless you attach two further conditions discussed in the following Remark.

Remark 7: We know that almost all real numbers are irrational. (Here, the term "almost all" means that all reals that are not irrational have measure zero).

Trying to understand why almost all infinite series with real number terms and almost all infinite series with rational number terms would be irrational caused me to realize the new method of theorem 4/theorem 6.

Part $7-1$:

Let us define an arithmetic function to mean a mathematical expression in variable n which involves the operators "+", "−", "×", "/" (division), and "**" (power) and uses these operators a finite number of times. It is not necessary for us to define arithmetic functions strictly in this manner. You can define an arithmetic function $f(n)$ differently, but it should be a defined as a *mathematical expression* in n, and not a "conditional" function.

$f(n)=k$, where $k=\min\{3-f(n-1),\frac{1}{n^2}\}$ OR $f(n)=3$ for n odd, $f(n)=n^2$ for n even OR other such definitions which are not direct mathematical expressions are not considered arithmetic functions. In the first paragraph of this paper are three examples of valid arithmetic functions.

If you find this definition of arithmetic functions to be ambiguous then define arithmetic functions to just mean functions of type $\frac{p(n)}{q(n)}$, where p and q are polynomials of n with rational coefficients. This limitation of the first definition will not cause any problems with the arguments that follow, because what holds for the broader definition will hold for this definition. If $f(n)$ is an arithmetic function then the below conditions can be added to theorem 6 (or added to theorem 4 if "$F(n)$" is replaced by "$f_k(n)$ for each $k,(1\le k\le j)$".

(4) $F(n)$ must be expressible in a form other than an infinite series.

(5) $\lim_{n\to\infty}F(n)$ must be expressible in a form other than an infinite series.

"Expressible in a form other than an infinite series" means that when we write the expression for $F(n)$ or for $\lim_{n\to\infty}F(n)$, the expression must not contain an infinite series.

It is worth noting that conditions (4) and (5) cannot be expressed in a different way or a more "preferable" way. I have expressed them in the only way possible, and they are perfectly well-suited, as they are stated, to result in theorems such as theorem 2. There is no ambiguity in these conditions. Consider, say, condition (4). If it is clear what an infinite series is then given a $F(n)$ that satisfies conditions (1) through (3) we can select which $F(n)$ are infinite series. The ones *not* selected satisfy condition (4).

The above conditions (4) and (5) can be applied to the following examples:

$F(n)=(\frac{4}{7}-\sum_{k=1}^{n-1}\frac{1}{k(k+1)})$ is invalid because $\lim_{n\to\infty}F(n)$ contains an infinite series

However, using partial fractions cancellation, we can re-write the above function to get $F(n)=(\frac{4}{7}-1+\frac{1}{n})$ which is valid.

$F(n)=(\frac{1}{n^2}+\sum_{k=n}^{\infty}\frac{1}{k!})$ is invalid because $F(n)$ contains an infinite series.

Taken with these conditions what do theorem 4/theorem 6 tell us? What they state is that, in all series with rational terms that converge to a rational number, $f(n)$ can be broken into parts which cancel each other out. For example in theorem 6, we get $\sum_{n=1}^{\infty}f(n)=([F(1)-F(2)]+[F(2)-F(3)]+\cdots+[F(n)-F(n+$

$1)]+[F(n+1)-F(n+2)]+\cdots)=F(1)$. Theorem 4 will give a cancellation pattern which involves breaking into more parts, but is similar.

Choose $F(n)=(S-\sum_{k=1}^{n-1}f(k))$, where S is some rational number. This would satisfy conditions (1) and (2) of theorem 6; but it will satisfy (3) of theorem 6 if and only if, for some rational number S, $(S-\sum_{k=1}^{\infty}f(k))=0$. But in theorem 6 we are using conditions $(1),(2)$ and (3) to decide whether $\sum_{k=1}^{\infty}f(k))=S$ for some rational number S. So with this choice of $F(n)$ theorem 6 will be totally circular because we cannot check condition (3) until we know whether $\sum_{k=1}^{\infty}f(k))=S$ for some rational number S.

Similarly, choosing $F(n)=(\sum_{k=n}^{\infty}f(k))$ would satisfy conditions (2) and (3) but in deciding condition (1) we would again go into a circular loop. The added conditions (4) and (5) serve to eliminate choices such as the last two where $F(n)$ is a circular choice; these two choices of $F(n)$ are "fake" choices because $f(n)$ is not being "actually" broken into two parts, even though we have $f(n)=F(n)-F(n+1)$. Thus we see that without the above conditions (4) and (5) theorem 4 and theorem 6 would be unusable and would just be circular statements. However, with these conditions theorem 4/theorem 6 become powerful statements which can be used to examine the question of convergence to rational or irrational numbers of any infinite series whose terms are defined by some arithmetic function which is rational for all n. To sum up, if $f(n)$ is rational for all n and the series converges to a rational number then $F(n)=(S-\sum_{k=1}^{n-1}f(k))$ or $F(n)=(\sum_{k=n}^{\infty}f(k))$ will collapse down, because of some cancellation pattern, to a $F(n)$ that satisfies all conditions (1) through (5). If this $F(n)$ remains as a infinite or finite series then this $F(n)$ would violate conditions (4) and (5), respectively. So when the series converges to a rational number, $f(n)$ must be broken into parts which form an "actual" cancellation pattern, with the conditions (4) and (5) serving as tests to insure that $F(n)$ is not circular, "fake" cancellation.

The F we choose may not conform to our definition of arithmetic function (especially if you choose the second, more restrictive definition) but still must be some sort of regular *mathematical expression* since $F(n)=(S-\sum_{k=1}^{n-1}f(k))$, and f is an arithmetic function. Conditions (4) and (5) follow logically from theorem 6/ theorem 4, and therefore are just observations of the nature of $F(n)$ that is implied by theorem 6/theorem 4. How these limitations on $F(n)$ are implied by theorem 6/theorem 4 is explained below.

Instead of conditions (4) and (5) we could say that $F(n)$ must be a "closed expression" or a "closed form"; instead of (4) and (5) we could also say that the infinite series $\sum_{n=1}^{\infty}f(n)$ is a "telescoping series".

The observation that conditions (4) and (5) must hold is what makes theorem 4/theorem 6 a workable test of rationality. The below arguments explain why it is justified to add conditions (4) and (5) to theorem 6/theorem 4.

Part $7-2$:

Let us begin by looking at a finite series of, say, 1000 terms all of which are positive rational numbers. In order to make this finite series converge to a rational number do the terms $f(1), \dots, f(1000)$ have to be chosen in a way such that our choice of $f(n+1)$ depends on our choice of any of the previous terms $f(1), \dots, f(n)$. Clearly not; we can choose each term independently (i.e. without worrying about what the other terms are) and the finite series will converge to a rational number.

Now suppose we want the finite series to converge to $\frac{4}{7}$. We could choose 999 terms to be any rational number less than, say $\frac{4}{7000}$. And then choose $f(1000)$ after seeing what $\sum_{k=1}^{999} f(k)$ is. To make a finite series converging to $\frac{4}{7}$ we cannot choose *all* the terms *independently* of each other; in our above way of constructing the finite series $f(1000)$ was chosen *depending* on what the others terms added to.

Now consider an infinite series converging to $\frac{4}{7}$. We will have $[\frac{4}{7} - f(1) - f(2) - \cdots - f(N)] = [f(N+1) + f(N+2) + \cdots$ (sum to ∞)] for any N

Now for the question of rationality of the infinite series we would only be concerned that the right side be a rational number. But we are interested not that the right side be a rational number but why it would be a rational number that equals the number on the left side. Indeed in asking this question instead of the question of convergence of the right side to a rational number — in asking this question we are inventing a new criterion that infinite series that converge to rational numbers must fulfill (as explained later the criterion would still be the same if instead of $\frac{4}{7}$ we had another rational number S). Note that we are interested only in what property f must have so as to have the above connection between the terms upto N (i.e. the terms on left) and the terms after N, no matter what N we choose; we are not investigating questions of rationality/irrationality of the right side. The right side must have some sort of "memory" of the left side because the right side must always equal not any rational number but the particular rational number on the left side. What is built into functions $f(n)$ that gives the later terms a "memory" of the previous terms?

Let us look at the matter in a new way. Suppose someone asked you to create an infinite series $\sum_{n=1}^{\infty} f(n)$ that converges to $\frac{4}{7}$. Let us say all terms of the series are positive and $f(n)$ has a rational value for all n. One would construct it this way: Let $f(1) = \frac{4}{7} - F(2)$.

Then $f(2) = F(2) - F(3)$

and $f(3) = F(3) - F(4)$

and similarly for any $n, f(n) = F(n) - F(n+1)$

When we make the infinite series by adding up the above terms a canceling chain of F is formed. (Theorem 4/theorem 6 can be proved in a simpler way. However, I have proved it in a way so that it best brings out the point that this canceling chain of F is formed.) The point is that the function f, in an infinite series that converges to $\frac{4}{7}$, is just the difference of successive terms of a sequence F of rational numbers. And you can choose any F which is rational for all integers n and $\lim_{n\to\infty} F(n) = 0$.

Now suppose we have to construct an infinite series $\sum_{n=1}^{\infty} f(n)$ as before but this time the function f should also be an arithmetic function. The F we choose may not conform to our definition of arithmetic function but, as discussed in part $7-1$, must still be some sort of regular *mathematical expression*. Thus to build such a infinite series with rational terms converging to $\frac{4}{7}$ we must choose some mathematical expression F so that $F(1) = \frac{4}{7}$ and $F(n)$ is rational for all n and $\lim_{n\to\infty} F(n) = 0$. From this chosen F we get f by putting $f(n) = F(n) - F(n+1)$. Now to say that our choice for F so as to build the required series is $F(n) = (S - \sum_{k=1}^{n-1} f(k))$ or is $F(n) = (\sum_{k=n}^{\infty} f(k))$ is a circular and logically invalid choice; (even though it is true the equalities $F(n) = (S - \sum_{k=1}^{n-1} f(k))$ and $F(n) = (\sum_{k=n}^{\infty} f(k))$ will hold once we have chosen suitable F that give us the required f that form the infinite series). The choice is invalid because the arithmetic expression f is obtained from some function F by putting $f(n) = F(n) - F(n+1)$. Now to go back in a circle and define F in terms of f is nonsense. Well, we could make F some infinite series which is known to converge to a rational number (by some other method) and say that using this F we can make an infinite series where $f(n) = F(n) - F(n+1)$. The circularity here is that F would be $F(n) = (\sum_{k=n}^{\infty} f(k))$ in order to give $f(n) = F(n) - F(n+1)$. And this this just the same situation — defining F in terms of f. Also the claim that we can get around the requirements of this method by choosing F as an infinite series that is known to converge to a rational number "by some other method" — this claim is nonsense because the method of this paper applies to all infinite series with rational terms — if this infinite series converges to a rational number then it must satisfy conditions (1) through (5). It should thus be clear that for infinite series, with rational terms which are arithmetic functions, converging to $\frac{4}{7}$ conditions (4) and (5) can be added.

But we are not interested in conditions for an infinite series converging to the fixed rational number $\frac{4}{7}$, but converging to *any* rational number. As we have seen, for series with a finite number of terms we get a totally different situation when the finite series converges to $\frac{4}{7}$ and when it converges to any rational number.

But we never used the value to be specifically $\frac{4}{7}$. If we had any rational number S the above reasoning and conclusions would still hold.

It is interesting, however, to note that if the infinite series converges to an irrational number then the investigation of how the left side of the equation equals the right side stops as soon as we *define* some irrational number to be the sum of the infinite series. We can do this since irrational numbers can be defined and brought into existence by being designated to be sums of certain series. Suppose we have some infinite series. We can investigate (in the way of remark $7-2$) how it could go to $\frac{4}{7}$. Suppose it does not go to a rational number (but converges). Then we can just say that let us call the irrational number it goes to by some name, say "qi". If we make such an assignment then it makes no sense to have an investigation of what special properties the terms of the series must have so that to go to the number "qi". So we cannot even begin to ask the questions we asked when the series went to the rational number $\frac{4}{7}$. This is because we would be asking why it goes to "qi" and not to some other name. Also, it does not matter if we had an irrational number like π which also has other properties. In that case we would have an irrational number that satisfies some properties and also it can be shown to be represented by some infinite series. We call this irrational number by the name "π"; and then the same logic holds. Arguing that the name "π" came first and it was later that mathematicians found an infinite series that equalled it does not change anything. We can get statements equivalent to conditions (1) through (3) (theorem 4/theorem 6) for infinite series going to irrational numbers, but not conditions (4) and (5). And this difference follows in so subtle a way that we really use no criterion for rationality. (Of course there can also be *some* series going to irrational numbers that give closed forms. But not all infinite series converging to irrational numbers will give closed forms.) Any real number which is not rational is *just a name given to the sum of a convergent series* – they are all just names. Consider that "square root of 2" is just a name of the convergent infinite series which has the property that its square is 2 – one could alternatively call it the "son of 2." Of course one could call $\frac{4}{7}$ the"son of $\frac{16}{49}$" but it will still be $\frac{4}{7}$ and we can ask why a series converges to this particular number – because we have something more than a name.

This ties in with Georg Cantor's showing that almost all real numbers are irrational. Infinite series have to have a very special property to converge to a rational number and my statements imply that almost all infinite series converging will converge to irrational. So this is an independent way to come to Cantor's realizations that rational numbers have measure zero.

I am a great admirer of Cantor (of course, his ideas are now universally acknowledged to be revolutionary) and it was very tragic that Leopold Kronecker blocked and dismissed Cantor, saying his work was simply "not mathematics." But Kronecker was, in a sense, absolutely correct when he said irrational "do not exist" and are the "work of man." *In a deeper sense* this is a central property of irrational numbers which has been exploited here to lead to new truths.

Another way to look at it is that when the infinite series goes to an irrational number then the terms can be chosen independently of each other, because as they go to infinity we have an irrational number which is a non-terminating, non-recurring decimal and so this number can just be defined to be whatever the terms add to — as we add more terms we get more decimals of this non-terminating decimal and this goes on forever; also the irrational number can be given a name "qi". If the infinite series goes to a rational number then the terms, which go on and on, have to be compatible with each other so as to go to a decimal which terminates or recurs.

The case of infinite series converging to an irrational number is very similar to the case of a finite series converging to a rational number. In the finite series case, the S can just be chosen to be whatever all the terms of the finite series add up to, just like the number "qi" can be chosen to be whatever the terms of the infinite series add to. It is worth noting again that for both infinite and finite series, with rational terms, converging to $\frac{4}{7}$ the $f(n)$ must have a memory of the sum of the previous terms, and you cannot choose each of the terms independently from each other. For infinite series converging to any rational number the f must still have this "memory"; however, for finite series converging to a rational number each of the terms may be independently chosen. In comparing finite series with rational terms converging to $\frac{4}{7}$ and infinite series with rational terms converging to any rational number one most note a major difference in that the infinite series requires the equality on p.6 to hold "for all N" (where N will run through all numbers, including an infinite number of primes etc) thus requiring a later (paired) cancelling term to also contain that same number. A finite series with rational terms converging to $\frac{4}{7}$ can go to this rational number by "coincidence" (for example consider the case of a series with only one term) and for finite series we cannot get theorems such as the ones we got for infinite series.

We said that conditions (4) and (5) hold if $f(n)$ is an arithmetic function, thus implying that they may not hold for other f. The reason for this becomes clear by considering the following example of making a series that converges to a rational number S:
Choose $f(1)$ to be any rational number between 0 and S.
Choose $f(2)$ to be a rational number between 0 and $S - f(1)$.
Choose $f(n)$ to be a rational number between 0 and $S - \sum_{k=1}^{n-1} f(k))$ and that also satisfies $S - \sum_{k=1}^{n} f(k) = 0$ as $n \to \infty$.
In this example the terms are such because each successive term $f(n)$ is chosen after considering what $S - \sum_{k=1}^{n-1} f(k))$ is.
However, if the terms are defined by some arithmetic function then the value of each term is fixed by the expression that defines $f(n)$ and cannot be chosen depending on other terms.

If my arguments above are unsatisfactory (perhaps because they do not look like mathematical proofs one generally reads), I hope they will provide the basic ideas using which you can make your own arguments to understand why conditions (4) and (5) follow. Conditions (4) and (5) follow logically – I can only explain the logic of why they follow. If you spend some time thinking about it the logic should become obvious. They may be subtle and but the logic is solid.

The reason for these arguments being the way they is is because equations, substitutions etc. which make up a normal mathematics proof won't do to prove something so general and basic. However, if you think that conditions (4) and (5) need not be true then one way you can *prove* them wrong is by finding a counter-example to theorem 2 or theorem 8, since these theorems are proved by using conditions (4) and (5). The above is a deep explanation and has to be thought about. In any case, we have to live with the fact that conditions (4) and (5) do hold in real life and I have realized that the only way to prove them is to come back to the arguments in part $7-2$. Trying totally different ways will be futile.

Remark 11**:** There are some series that are a little more challenging, even with these theorems. For example, theorem 4 cannot easily determine whether series that involve x^n, where x is a constant will be rational or irrational; this is because replacing $x^n = (y-z)^n$, where y and z are constants, will cause new possibilities of cancellation. I shall discuss some such individual series more fully in a future paper. I chose to illustrate the use of my approach for series whose terms are quotients of polynomials because in this case we are able to form a single test for this family of infinite series. For other series a case by case application of the theorems will generally be needed. To establish whether an infinite series is rational or irrational one simply has to establish whether the terms are compatible in such a way that they will cancel each other out. This should be the logical and standard way to decide rationality and irrationality of numbers for whom the infinite series equal, having terms expressed as a function which takes rational values for all n, is known. Using this method, it is fair to say that the irrationality of infinite series, infinite products, and various other infinite processes now becomes a matter of routine labor rather than a serious intellectual challenge requiring original investigations in number theory.

We can also use the ideas of these methods to analyze infinite series, and learn facts about them other than what kind of real number they converge to. To illustrate this we solve the following problem quoted from a 1826 paper by Neils H. Abel [2]:

"One of the most remarkable series of algebraic analyses is the following: $1+\frac{m}{1}x+\frac{m(m-1)}{1\cdot 2}x^2+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3+\cdots+\frac{m(m-1)\cdots[m-(n-1)]}{1\cdot 2\cdots n}x^n+\cdots$

When m is a positive whole number the sum of the series, which is then finite, can be expressed, as is known, by $(1+x)^m$. When m is not an integer, the series goes on to infinity, and it will converge or diverge according as the quantities m and x have this or that value. In this case one writes the same equality

$(1+x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{1\cdot 2}x^2 + \cdots$ etc.

$\cdots$ It is assumed that the numerical equality will always occur whenever the series is convergent, but this has never yet been proved."

Now an alternate way to prove this would be that if we knew the infinite series converges to a rational number then there must exist a cancellation pattern. A search for a cancellation pattern, say for the case $m = \frac{1}{2}$ leads to replacing x by $y^2 - 1$. Also further replace $y^2 - 1 = (y-1)(y+1)$ by $z(z-2)$, putting $z = 1 - y$. In each term replacing x with $z^2 - 2z$, expanding each term, and adding the same powers of z from all terms, gives us the required cancellation pattern. When $m = \frac{1}{3}$ we would begin by replacing x with $y^3 - 1$ etc. Thus this numerical equality is proved by finding and using this cancellation pattern, without actually using any of the theorems in this paper. This example is unusual because here we have a cancellation pattern for both cases — when the series converges to a rational or to an irrational number; this kind of occurrence is what we can often expect when the terms involve x^n and the series converges to a rational number for some values of x. But what led us to this easy solution was that we knew that because for various values of x the series converges to a rational number there must be a hidden cancellation pattern.

Remark 12**:** Though this paper deals with classifying real numbers into rational and irrational we can, along the same lines, develop theorems about other kinds of numbers. The basic idea that leads to the above theorems can be directly applied to infinite processes involving numbers that satisfy some property, to determine whether the number the infinite process converges to will satisfy the same property. The major deciding factor will be how the numbers are defined.

As mentioned in remark 7, because irrational numbers are defined in a *negative* manner, (every number which is *not* a quotient of integers is called irrational), we cannot have an algorithm such as Theorem 2 that will decide whether an infinite series with all irrational terms will converge to an irrational.

So this paper is as much about number theory as about abstract algebra since we never used any properties as such of rational numbers or irrational numbers. Fields constructed similar to the way rational numbers are constructed will give similar theorems. The expression of this paper into formal algebraic terms should be possible.

REFERENCES

1. Apéry, R., *Irrationalité de* $\zeta(2)$ *et* $\zeta(2)$ Astérisque 61, 11-13, 1979; A. van der Poorten, *A Proof that Euler missed ... Apéry's proof of the irrationality of* $\zeta(3)$, Math. Intelligencer 1 (1978/79), no. $4, 195-203$

2. N.H.Abel, *Untersuchungen über die Reihe* $1+\frac{m}{1}x+\frac{m(m-1)}{1\cdot 2}x^2+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3+\cdots$, Journal für die reine und angewandte Mathematik (1826) (Reprinted in *Ouvres Completès d Abel*; Christiana. Johnson Reprint Corporation, New York, 1965)